The Stochastic heat and wave equation

with irregular noise coefficients: well-posedness, long time behavior and small mass limit

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- "*On quasi-linear stochastic partial differential equations", Gyöngy and Pardoux,1993.*

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- Weak uniqueness: X_t is a time changed Brownian motion.
- Strong uniqueness: imposing Sobolev regularity conditions on σ and use PDE theory.
- None of them works in infinite dimensions.

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$$W(dxdt) = \sum_{n=1}^{\infty} e_n(x)dB_t^{(n)}dx.$$

$$N(t,x) = \sum_{n=1}^{\infty} \int_0^t \int_{\mathbb{R}^d} \mathcal{S}(t-s, x-y)g(s,y)e_n(y)dydB_s^{(n)}.$$

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- Rough path theory, regularity structure, paracontrolled calculus.

$$\partial_t \Phi = \Delta\Phi + C\Phi - \Phi^3 + \xi$$

Only for additive noise or sufficiently regular noise coefficient.

Known results for the stochastic Heat equation

- Additive noise case $\partial_t u = \Delta u + f(u) + \frac{\partial^2 W}{\partial_t \partial_x}$.
  - Hilbert space valued SPDEs: strong solution for Hölder continuous $f(u)$. Strong solution for a.e. initial value for bounded measurable f. "*Strong uniqueness for SDEs in Hilbert spaces with nonregular drift.*"

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  - Random field SPDEs: strong solutions for distributional $f(u)$. *"Well-posedness of stochastic heat equation with distributional drift and skew stochastic heat equation."*
- Multiplicative, Hölder continuous noise coefficient: σ being $\frac{3}{4} + \epsilon$-Hölder continuous in $X(t,x)$, the random field case:

$$\frac{\partial}{\partial_t} X(t,x) = \frac{1}{2} \Delta X(t,x) dt + \sigma(t,x,X(t,x)) dW(t,x)$$

  "Pathwise Uniqueness for Stochastic Heat Equations with Hölder Continuous Coefficients: the White Noise Case", 77 p".

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- B has linear growth.

Well-Posedness

Theorem (Well-posedness of Stochastic Heat equation)

Under the assumptions in the previous slide, there exists a unique (probabilistic weak) mild solution to

$$dX_t = AX_t dt + B(X_t)dt + \sigma(X_t)dW_t, \quad X_0 \in H. \tag{4}$$

Given $\frac{1}{2} + \epsilon$-Hölder $F : H \to H$, unique weak-mild solution to

$$dX_t = AX_t dt + (-A)^{1/2}F(X_t)dt + B(X_t)dt + \sigma(X_t)dW_t, \tag{5}$$

- Examples: Burgers type equations, $\xi \in (0, 2\pi)$

$$du(t,\xi) = \frac{\partial^2}{\partial \xi^2} u(t,\xi)dt + \frac{\partial}{\partial \xi} h(u(t,\xi))dt + \sigma(u(t,\xi))dW_t(\xi).$$

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- Cahn-Hilliard equations in dimensions 1,2,3:

$$du(t,\xi) = -\Delta_\xi^2 u(t,\xi)dt + \Delta_\xi h(u(t,\xi))dt + \sigma(u(t,\xi))dW_t(\xi)$$

Long-time behaviour

Theorem (Exponential ergodicity)

Assume the drift $B : H \to H$ is Hölder continuous, and the Lyapunov condition hold: for some $V : H \to \mathbb{R}_+$ and some $\lambda \in (0, 1)$ infinity at infinity,

$$\mathbb{E}[V(X_t)] \leq \lambda V(X_0) + M \tag{6}$$

for some given $t > 0$ and $M > 0$. Then there exists a unique invariant measure, and the solution converges to the invariant measure exponentially fast with respect to (some specific) Wasserstein distance on $\mathcal{P}(H)$.

- No applicable Itô formula for cylindrical noise

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- Lyapunov assumption satisfied when B, F, σ are bounded, and A is a negative operator.

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- In order to compare X_t to a process

$$dX_t^n = AX_t^n dt + B^n(X_t^n)dt + \sigma^n(X_t^n)dW_t,$$

consider auxiliary process with $\lambda > 0$ and stopping time τ

$$d\widetilde{X}_t^n = A\widetilde{X}_t^n dt + B^n(\widetilde{X}_t^n)dt + \lambda 1_{t\leq\tau}dt + \sigma^n(\widetilde{X}_t^n)dW_t,$$

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- Probabilistic estimate: compare $X^n|_{[0,T]}$ and $\widetilde{X}^n|_{[0,T]}$ via Girsanov transform and Pinsker's inequality.
- Use pathwise estimate to compare $X|_{[0,T]}$ with $\widetilde{X}^n|_{[0,T]}$.

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Some technical challenges in infinite dimensions:

- Derive maximal inequality for the process, when λ is large:

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- No applicable Itô's formula: always work with mild formulations.
- Lipschitz approximation in infinite dimensions: compactness of heat semigroup.

Stochastic wave equation: well-posedness

Our method works not only for the parabolic systems, but also for hyperbolic systems.

Consider the (abstract) damped stochastic wave equation

$$\mu \frac{\partial^2 u_\mu(t)}{\partial t^2} = Au_\mu(t) - \frac{\partial u_\mu(t)}{\partial t} + B(t, u_\mu(t)) + G(t, u_\mu(t))dW_t,$$

and the stochastic wave equation without damping term

$$\mu \frac{\partial^2 u_\mu(t)}{\partial t^2} = Au_\mu(t) + B(t, u_\mu(t)) + G(t, u_\mu(t))dW_t,$$

Theorem (Well-posedness of stochastic wave equation)

Under the same assumption on A, B and G as in the case of the stochastic heat equation, there exists a unique weak-mild solution to both these equations.

Stochastic wave equation: small mass limit

Theorem

Assume moreover that B is Hölder continuous in u_μ. Then as μ tends to 0, the solution to the damped stochastic wave equation

$$\mu \frac{\partial^2 u_\mu(t)}{\partial t^2} = Au_\mu(t) - \frac{\partial u_\mu(t)}{\partial t} + B(t, u_\mu(t)) + G(t, u_\mu(t))dW_t$$

converges in distribution on path space to the solution of the stochastic heat equation

$$\frac{\partial u_\mu(t)}{\partial t} = Au_\mu(t) + B(t, u_\mu(t)) + G(t, u_\mu(t))dW_t.$$

"On the Smoluchowski-Kramers approximation for a system with an infinite number of degrees of freedom", Freidlin and Cerrai, 2006.

Thanks