# Scaling in Low-Dimensional Long-Range Percolation Models

Pierre-François Rodriguez

Imperial College London

Phase Transitions and Correlated Random Processes

Isaac Newton Institute, Cambridge, UK

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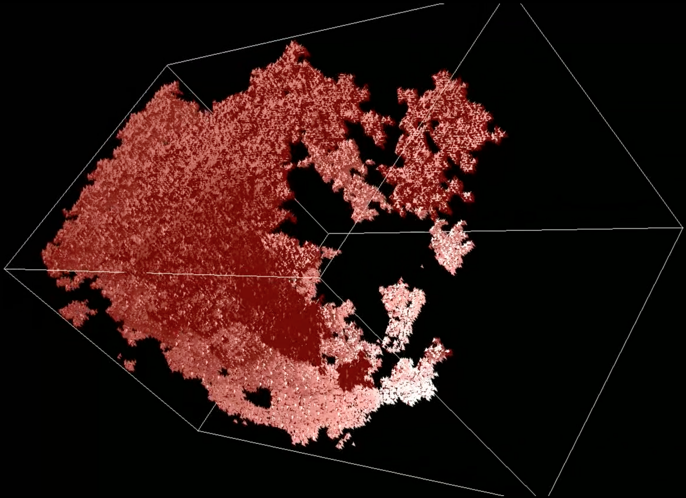
# Criticality



Credit: Laurent Goergen

# Fragmentation by a RW

Brummelhuis-Hilhorst '92, Benjamini-Sznitman '06

Side-length $= N \gg 1$ Time $= \frac{\#\text{steps}}{N^3} = u > 0$

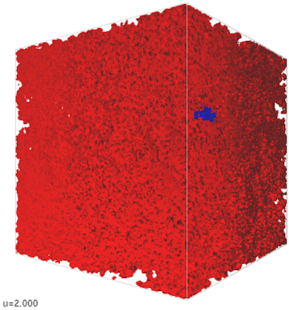



**Fig:** Vacant set of random walk. Largest (red) and $2^{nd}$ largest (blue) connected component. $N = 200$, $u = 2$.

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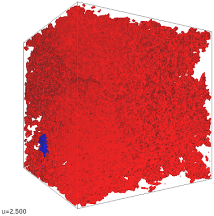
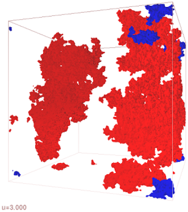
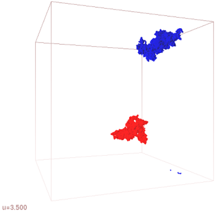


$u = 2.5$



$u = 3.0$

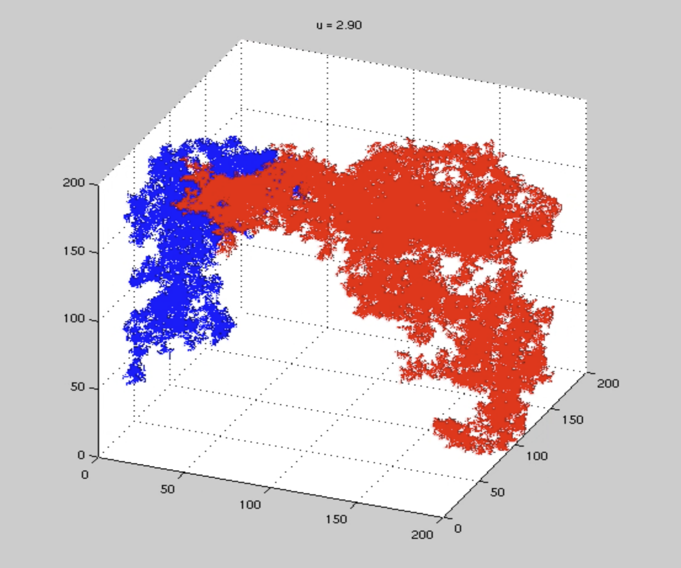


$u = 3.5$

Phase transition around $u_c \approx 3$ !

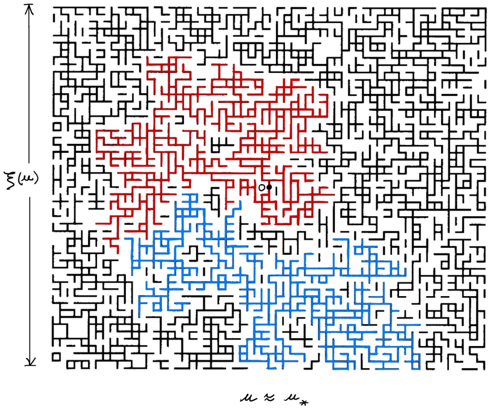
Simulations by D. Windisch.

# Near $u_c$



# Near $u_c$: heuristics

$$\tau_u(r) \stackrel{\text{def.}}{=} \mathbb{P}_u[0 \longleftrightarrow \partial B_r,\ 0 \not\longleftrightarrow \infty]$$




Scaling hypothesis:

$$\tau_{u_c}(r) \sim r^{-\frac{1}{\rho}},\ r \to \infty,$$
$$\tau_u(r) \asymp \tau_{u_c}(r) g(\tfrac{r}{\xi}),$$

where $g \in L^1$,

$$\xi = \xi_u = |u - u_c|^{-\nu}.$$

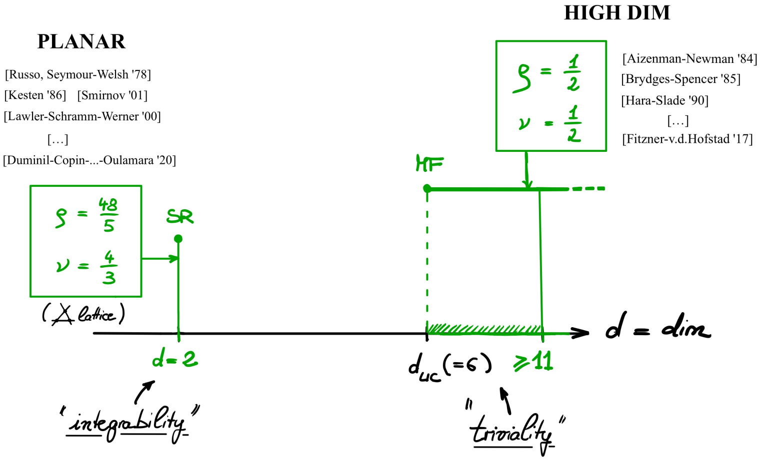
$\rho, \nu, \ldots$: critical exponents

Conj.: $\nu, \rho$, etc. **exist + are universal.**

+ satisfy (hyper-)scaling relations.

# Known vs. Unknown

(SR) Short-range: $\mathrm{Cov}(x, y) = 0, x \neq y$



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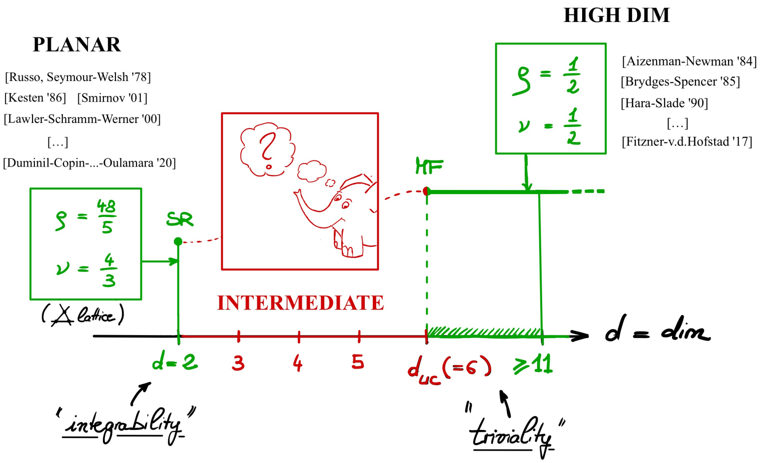
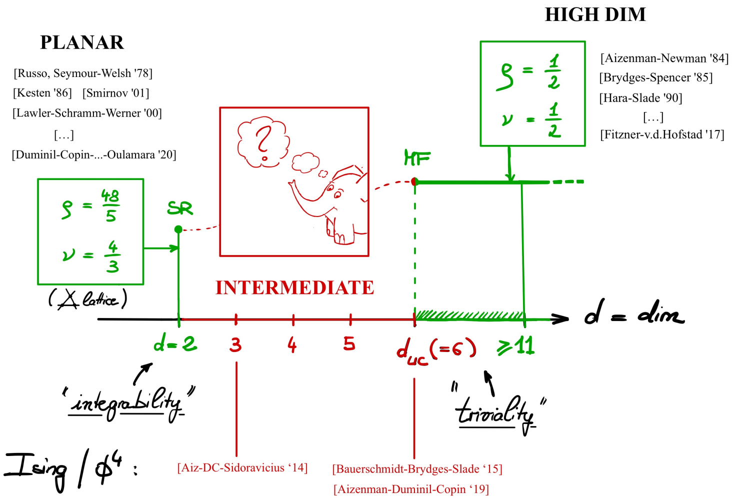


# Known vs. Unknown

(SR) Short-range: $\mathrm{Cov}(x, y) \approx 0, x \neq y$



# Paradigm shift

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(LR$_a$) Long-range: $\mathrm{Cov}(x, y) \sim |x - y|^{-a}, \; 0 < a < d.$

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Model: [Lebowitz-Saleur '86], [R.-Sznitman '12] .... [Černý '23], [Conchon-K. '23]

$\varphi = (\varphi_x)_{x \in \mathbb{Z}^d}$ Gaussian free field (GFF)

$$\mathbb{E}[\varphi_x \varphi_y] = (-\Delta_{\mathbb{Z}^d})^{-1}(x,y) \ : \text{class } (\mathsf{LR}_{d-2})$$

$u \in \mathbb{R}$: height.

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Percolation (à la [Lupu '16]): given $\varphi$ and $u$, keep $\{x,y\}$ indep. w. prob.

$$1-\exp\big\{-(\varphi_x-u)_+(\varphi_y-u)_+\big\}.$$

$C_u$: cluster of $0$ (at height $u$)

$$u_c = \inf\{u : \mathbb{P}[C_u \text{ unbounded}] = 0\}.$$

# Results [Drewitz-Prévost-R. '22, '23+]

Theorem

i) (Continuity, $d \geq 3$).

$$\mathbb{P}\text{-a.s.} \quad \mathsf{cap}(C_{u_c}) < +\infty$$

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ii) (Scaling, $d = 3$).

$$\tau_u(r) \stackrel{\text{def.}}{=} \mathbb{P}[0 \stackrel{C_u}{\longleftrightarrow} \partial B_r, 0 \stackrel{C_u}{\not\longleftrightarrow} \infty]$$

$$\asymp r^{-\frac{1}{\rho}} \exp\left\{-\frac{(r/\xi_u)}{1 \vee \log(r/\xi_u)}\right\}, \; u \in \mathbb{R}, \; r \geq 1^*,$$

where $\xi_u \propto |u - u_c|^{-\nu}$,

$$\boxed{\nu = \rho = \frac{2}{d-2}}.$$

* **UB:** all $u, r$. **LB:** all $u, r$ s.t. $\frac{r}{\xi_u} \leq 1$ or $f(\frac{r}{\xi_u}) \geq 1$, $f(x) = o(x \log x)$, $x \to \infty$.

# Continuity

$u < 0$: soft [Bricmont-Leb.-Maes '87]

Lemma

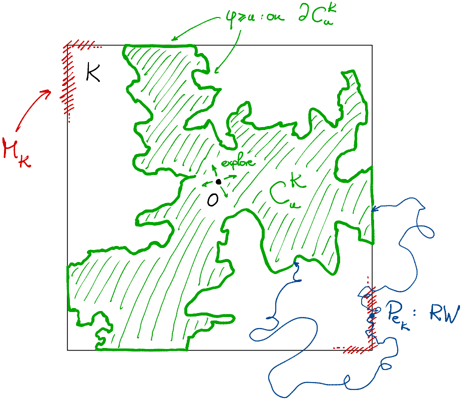
$C_u^K$: cluster of $0$ in $\{\varphi \geq u\} \cap K$

$$-\frac{d}{du}\mathbb{E}\left[F(C_u^K)\right] = \mathbb{E}\left[M_K F(C_u^K)\right]$$

where

$$M_K = \langle e_K, \varphi \rangle$$

$e_K$: equ. measure of $K$.




Markov:

$$\mathbb{E}\left[\, M_K \,\middle|\, \text{“}\varphi|_{C_u^K}\text{”}\right] = E_{e_K}\left[\varphi(X_{H_{C_u^K}})1\{H_{C_u^K} < \infty\}\right]$$

$$\geq u P_{e_K}[H_{C_u^K} < \infty] \overset{\text{sweep}}{\geq} u\mathrm{cap}\left(C_u^K\right) \; !$$

# Universality

All results true if: $G$ graph

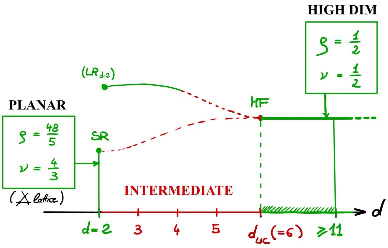
Vol. growth (geometric): $B_G(x,R) \asymp R^d$

Corr. decay (analytic): $(-\Delta_G)^{-1}(x,y) \asymp d_G(x,y)^{-a}, \quad a \leq 1(<2?)$

| Crit. Exp. | $\alpha$ | $\beta$ | $\gamma$ | $\delta$[2] | $\Delta$ | $\rho$ | $\nu$[1] | $\eta$ | $\kappa$ |
|---|---|---|---|---|---|---|---|---|---|
| Value | $2-\frac{2d}{a}$ | $1$ | $\frac{2(d-a)}{a}$ | $\frac{2d}{a}-1$ | $\frac{2d}{a}-1$ | $\frac{2}{a}$ | $\frac{2}{a}$ | $a-d+2$ | $\frac{1}{2}$ |

[1] Halperin-Weinrib: Phys. Rev. '83
[2] Sak: Phys. Rev. '73+'77

- all scaling relations hold
- exps. rational in $a$ and $d$
- $a = d-2$, $d \nearrow 6$ : MF !



Thank you !