

Scaling in Low-Dimensional Long-Range Percolation Models

Pierre-François Rodriguez

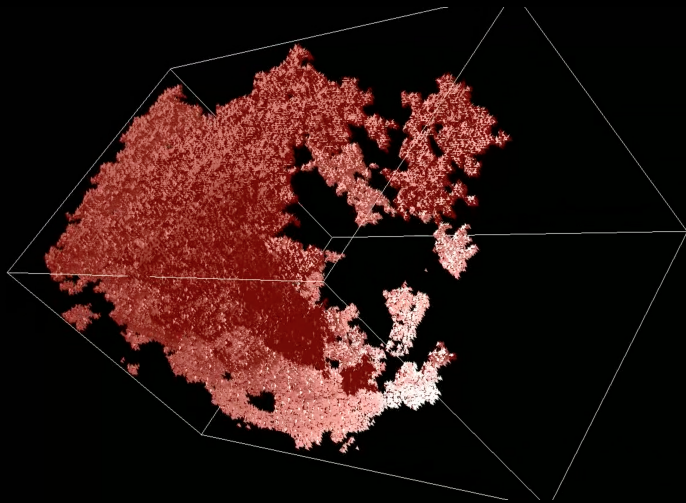
Imperial College London

Phase Transitions and Correlated Random Processes

Isaac Newton Institute, Cambridge, UK

May 10, 2023

Criticality



Credit: Laurent Goergen

Fragmentation by a RW

Brummelhuis-Hilhorst '92, Benjamini-Sznitman '06

$$\text{Side-length} = N \gg 1 \quad \text{Time} = \frac{\#\text{steps}}{N^3} = u > 0$$

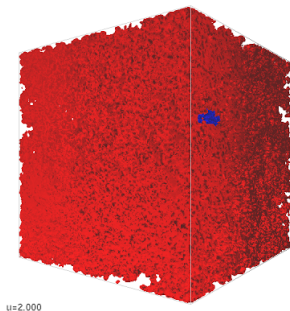
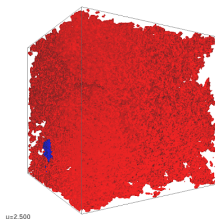


Fig: Vacant set of random walk. Largest (red) and 2nd largest (blue) connected component. $N = 200$, $u = 2$.

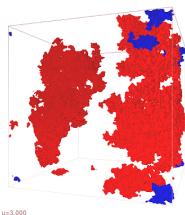
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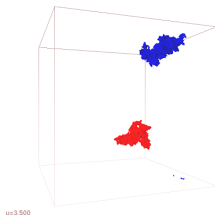
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$u = 2.5$



$u = 3.0$

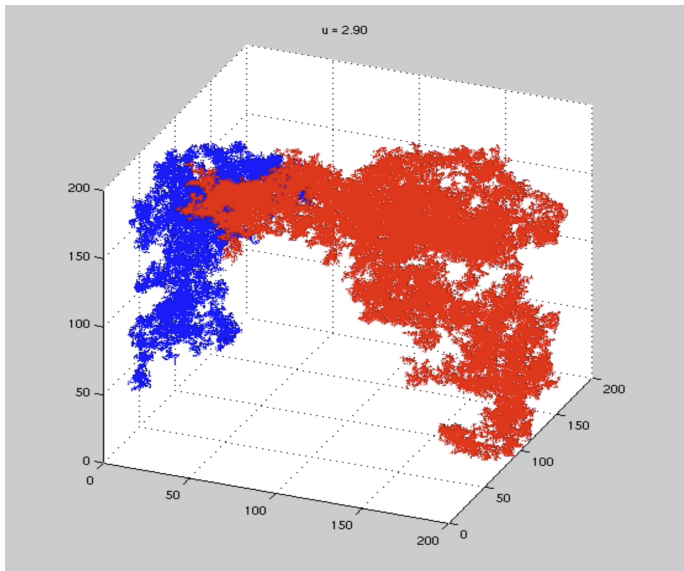


$u = 3.5$

Phase transition around $u_c \approx 3!$

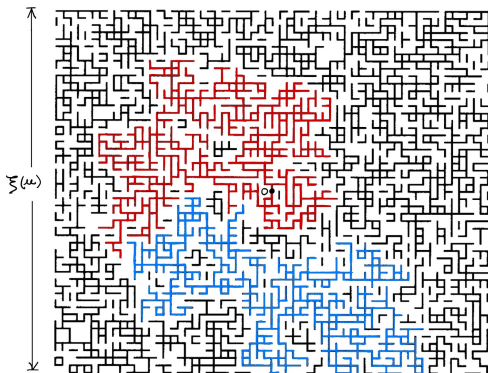
Simulations by D. Windisch.

Near u_c



Near u_c : heuristics

$$\tau_u(r) \stackrel{\text{def.}}{=} \mathbb{P}_u[0 \longleftrightarrow \partial B_r, 0 \not\longleftrightarrow \infty]$$



$$u \approx u_*$$

Scaling hypothesis:

$$\begin{aligned}\tau_{u_c}(r) &\sim r^{-\frac{1}{\rho}}, r \rightarrow \infty, \\ \tau_u(r) &\asymp \tau_{u_c}(r) g\left(\frac{r}{\xi}\right),\end{aligned}$$

where $g \in L^1$,

$$\xi = \xi_u = |u - u_c|^{-\nu}.$$

ρ, ν, \dots : critical exponents

Conj.: ν, ρ , etc. exist + are universal.

+ satisfy (hyper-)scaling relations.

Known vs. Unknown

(SR) Short-range: $\text{Cov}(x, y) = 0, x \neq y$

PLANAR

[Russo, Seymour-Welsh '78]

[Kesten '86] [Smirnov '01]

[Lawler-Schramm-Werner '00]

[...]

[Duminil-Copin-...-Oulamara '20]

$$\xi = \frac{48}{5}$$
$$\nu = \frac{4}{3}$$

(~~X~~ lattice)

SR

$d = 2$

"integrability"

HIGH DIM

$$\xi = \frac{1}{2}$$
$$\nu = \frac{1}{2}$$

[Aizenman-Newman '84]

[Brydges-Spencer '85]

[Hara-Slade '90]

[...]

[Fitzner-v.d.Hofstad '17]

HF

$d_{uc} (= 6) \geq 11$

"triviality"

$d = \text{dim}$

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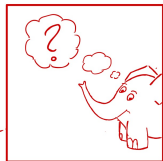
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$$\begin{aligned} \beta &= \frac{48}{5} \\ \nu &= \frac{4}{3} \end{aligned}$$

(\mathbb{Z} lattice)

SR



INTERMEDIATE

HF

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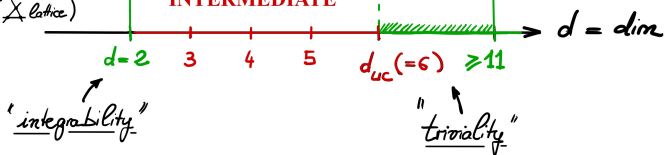
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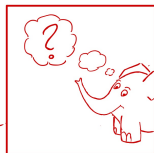
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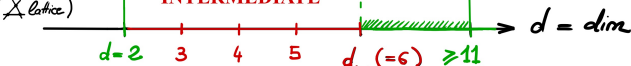
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MF



'integrability'

"triviality"

Ising / ϕ^4 :

[Aiz-DC-Sidoravicius '14]

[Bauerschmidt-Brydges-Slade '15]

[Aizenman-Duminil-Copin '19]

Paradigm shift

(LR_a) Long-range: $\text{Cov}(x, y) \sim |x - y|^{-a}$, $0 < a < d$.

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Model: [Lebowitz-Saleur '86], [R.-Sznitman '12] ... [Černý '23], [Conchon-K. '23]

$\varphi = (\varphi_x)_{x \in \mathbb{Z}^d}$ Gaussian free field (GFF)

$$\mathbb{E}[\varphi_x \varphi_y] = (-\Delta_{\mathbb{Z}^d})^{-1}(x, y) \quad : \text{ class (LR}_{d-2}\text{)}$$

$u \in \mathbb{R}$: height.

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Percolation (à la [Lupu '16]): given φ and u , keep $\{x, y\}$ indep. w. prob.

$$1 - \exp \left\{ -(\varphi_x - u)_+(\varphi_y - u)_+ \right\}.$$

C_u : cluster of 0 (at height u)

$$u_c = \inf \{ u : \mathbb{P}[C_u \text{ unbounded}] = 0 \}.$$

Results [Drewitz-Prévost-R. '22, '23+]

Theorem

i) (Continuity, $d \geq 3$).

$$\mathbb{P}\text{-a.s. } \text{cap}(C_{u_c}) < +\infty$$

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ii) (Scaling, $d = 3$).

$$\begin{aligned} \tau_u(r) &\stackrel{\text{def.}}{=} \mathbb{P}[0 \xleftrightarrow{C_u} \partial B_r, 0 \not\xleftrightarrow{C_u} \infty] \\ &\asymp r^{-\frac{1}{\rho}} \exp\left\{-\frac{(r/\xi_u)}{1 \vee \log(r/\xi_u)}\right\}, \quad u \in \mathbb{R}, r \geq 1^*, \end{aligned}$$

where $\xi_u \propto |u - u_c|^{-\nu}$,

$$\nu = \rho = \frac{2}{d-2}.$$

* **UB**: all u, r . **LB**: all u, r s.t. $\frac{r}{\xi_u} \leq 1$ or $f(\frac{r}{\xi_u}) \geq 1$, $f(x) = o(x \log x)$, $x \rightarrow \infty$.

Continuity

$u < 0$: soft [Bricmont-Leb.-Maes '87]

Lemma

C_u^K : cluster of 0 in $\{\varphi \geq u\} \cap K$

$$-\frac{d}{du} \mathbb{E}[F(C_u^K)] = \mathbb{E}[M_K F(C_u^K)]$$

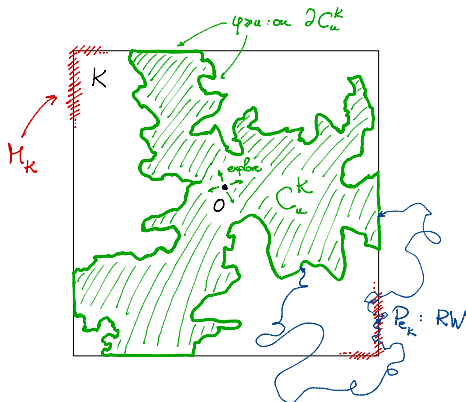
where

$$M_K = \langle e_K, \varphi \rangle$$

e_K : equ. measure of K .

Markov:

$$\begin{aligned} \mathbb{E}[M_K \mid \text{"}\varphi|_{C_u^K}\text{"}] &= E_{e_K}[\varphi(X_{H_{C_u^K}}) 1\{H_{C_u^K} < \infty\}] \\ &\geq u P_{e_K}[H_{C_u^K} < \infty] \stackrel{\text{sweep}}{\geq} u \text{cap}(C_u^K) ! \end{aligned}$$



Universality

All results true if: G graph

Vol. growth (geometric): $B_G(x, R) \asymp R^d$

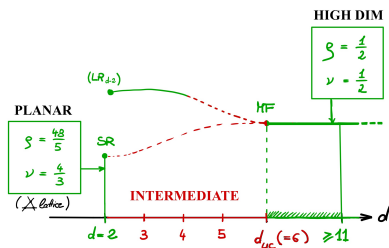
Corr. decay (analytic): $(-\Delta_G)^{-1}(x, y) \asymp d_G(x, y)^{-a}$, $a \leq 1 (< ??)$

Crit. Exp.	α	β	γ	δ^2	Δ	ρ	ν^1	η	κ
Value	$2 - \frac{2d}{a}$	1	$\frac{2(d-a)}{a}$	$\frac{2d}{a} - 1$	$\frac{2d}{a} - 1$	$\frac{2}{a}$	$\frac{2}{a}$	$a - d + 2$	$\frac{1}{2}$

¹Halperin-Weinrib: Phys. Rev. '83

²Sak: Phys. Rev. '73+'77

- all scaling relations hold
- exps. rational in a and d
- $a = d - 2$, $d \nearrow 6$: MF !



Thank you !