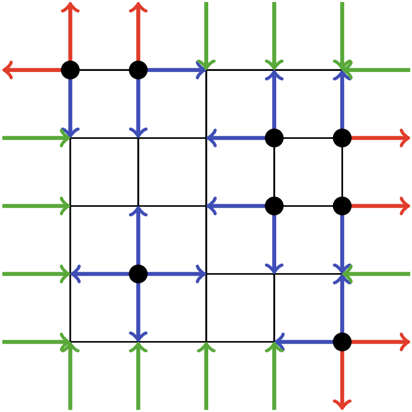
Universal cutoff for exclusion with reservoirs

Justin Salez (Université Paris-Dauphine)



1. Model and questions
2. Results
3. Proof ingredients

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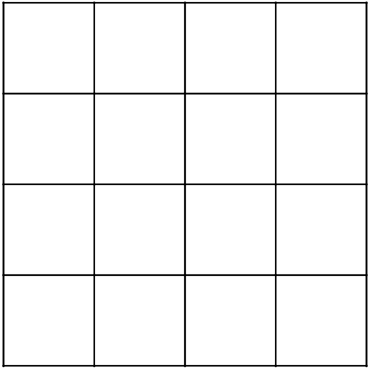

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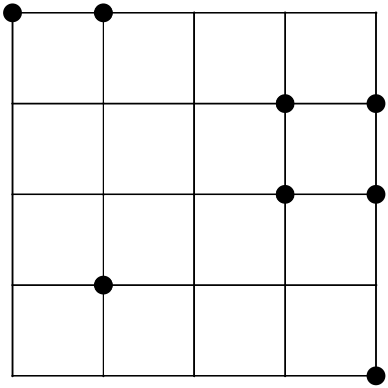

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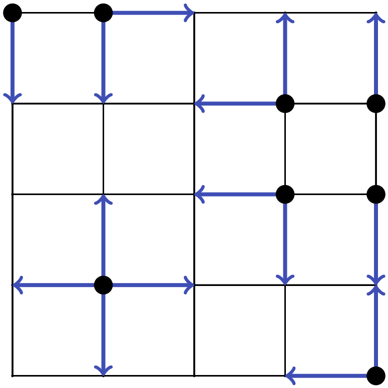

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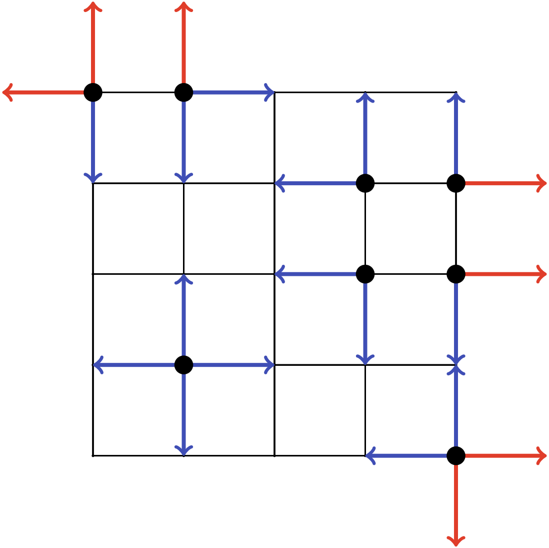

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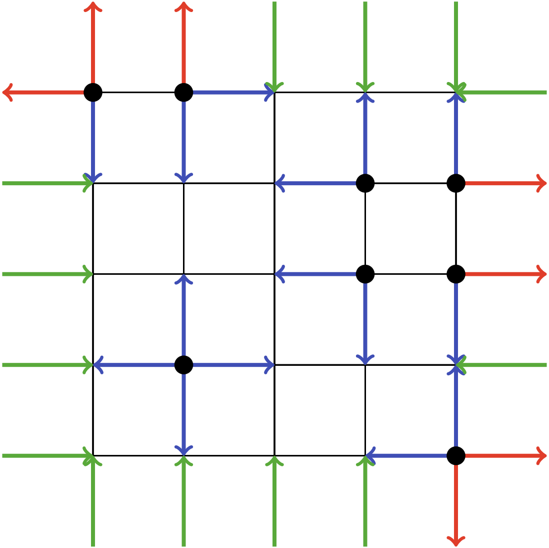

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Question: how fast?

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Goal: estimate $\mathrm{t}_{\mathrm{MIX}}(\varepsilon)$ when $\varepsilon \in (0,1)$ is fixed and $|\mathscr{X}| \gg 1$.

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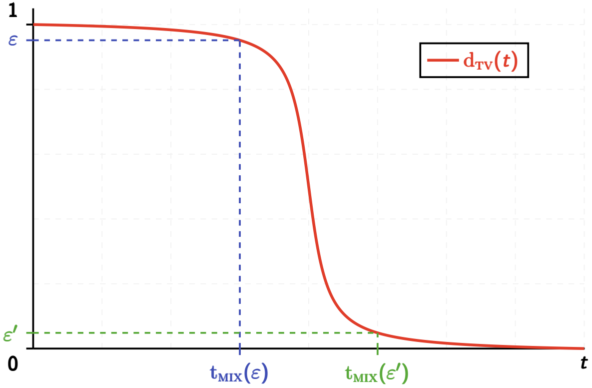

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- Can we go beyond the one-dimensional case?

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▷ Mixing occurs precisely when $\Psi(t)$ becomes of order 1

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▷ Non-conservative analogue of Aldous' spectral gap conjecture, famously proved by Caputo, Liggett & Richthammer (2010).

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- Insufficient in general (even for Abelian random walks...)
- Sufficient for birth-death chains (Ding, Lubetzky & Peres '10) and for random walks on trees (Basu, Hermon & Peres '17).

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$$\frac{\log \langle \phi_1, \mathbf{1} \rangle^2 - c}{2\lambda_1} \quad \le \quad \mathrm{t}_{\mathrm{MIX}}(\varepsilon) \quad \le \quad \frac{\log n + c}{2\lambda_1}.$$

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▷ Cutoff at time $\frac{\log n}{2\lambda_1}$ as soon as ϕ_1 is **delocalized**.

Consequence 3: mixing-time estimates

Recall that $\mathrm{d}_{\mathrm{TV}}(t)$ is controlled in a two-sided way by the quantity

$$\Psi(t) \;=\; \sum_{k=1}^{n} e^{-2\lambda_k t} \langle \phi_k, \mathbf{1} \rangle^2.$$

The latter is roughly governed by the **Perron eigenpair** (λ_1, ϕ_1).

Corollary 3: there is a universal constant $c = c(\varepsilon, \rho)$ such that

$$\frac{\log \langle \phi_1, \mathbf{1} \rangle^2 - c}{2\lambda_1} \;\leq\; \mathrm{t}_{\mathrm{MIX}}(\varepsilon) \;\leq\; \frac{\log n + c}{2\lambda_1}.$$

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▷ The upper-bound is a huge conjecture in the conservative case...

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1. Model and questions

2. Results

3. Proof ingredients

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▷ But whether Z is **localized**/**delocalized** should also play a role!

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Unfortunately, estimating $\left\| \frac{\mu}{\pi} - 1 \right\|_{L^2_\pi}$ is hard beyond product measures (c.f. "information percolation" by Lubetzky & Sly).

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Lemma: suppose Z satisfies the **negative dependence** condition

$$\forall A \subseteq [n], \qquad \mathbb{E}\left[\prod_{i \in A} Z_i\right] \leq \prod_{i \in A} \mathbb{E}\left[Z_i\right].$$

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Conclusion: μ is close to π if and only if $\sum_{i=1}^n \mathbb{E}^2[Z_i]$ is small.

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Corollary: our main estimate follows immediately!

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- Conservative case? (1D by Lacoin, 2016)

Thanks!