Characterising the Gaussian free field

Phase Transitions and Correlated Processes, Isaac Newton Institute
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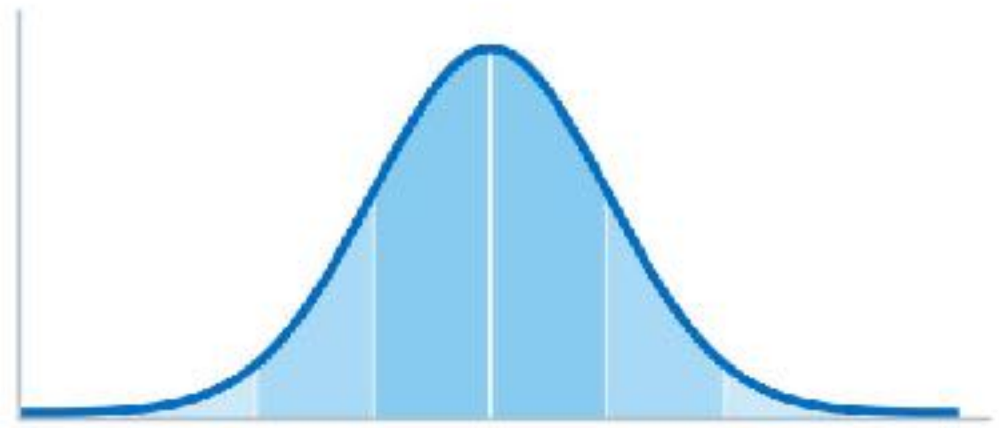
Ellen Powell, Durham University. Based on joint work with Juhan Aru, Nathanaël Berestycki and Gourab Ray.

Gaussian free field: Motivation

- **Gaussian/normal distribution**

$$Z \overset{(d)}{=} \lim_{n\to\infty} n^{-1/2} \sum_{i=1}^{n} X_i \text{ with } (X_i)_{i\geq 1} \text{ i.i.d. (centered, finite variance)}$$



Universal random variable

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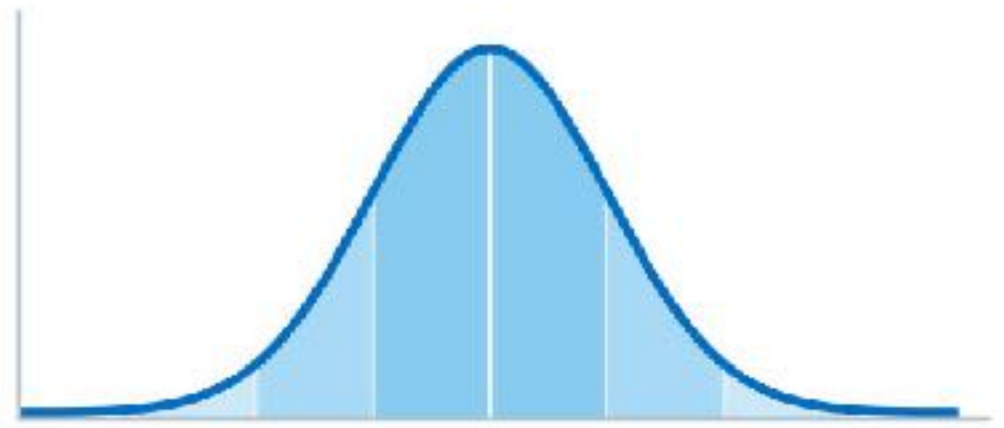

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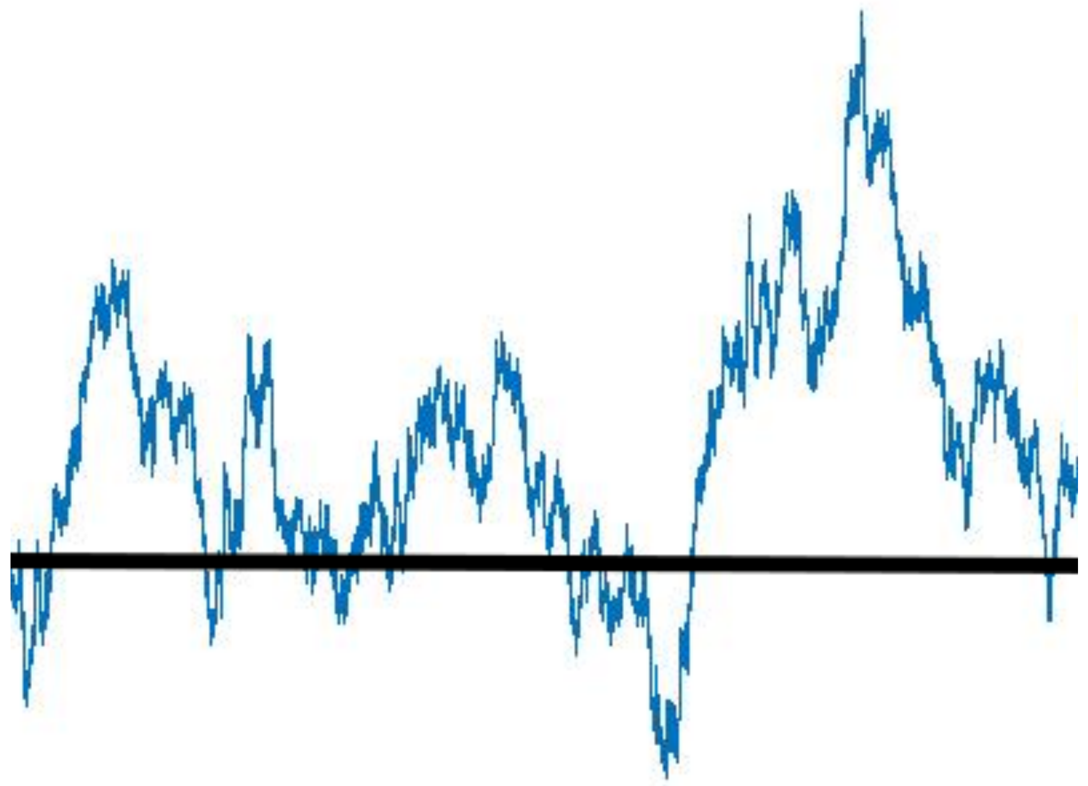

$$\left(B_t\right)_{t\geq 0} \overset{(d)}{=} \lim_{n\to\infty} \left(n^{-1/2} \sum_{i=1}^{\lfloor nt \rfloor} X_i\right)_{t\geq 0}$$

Universal random path

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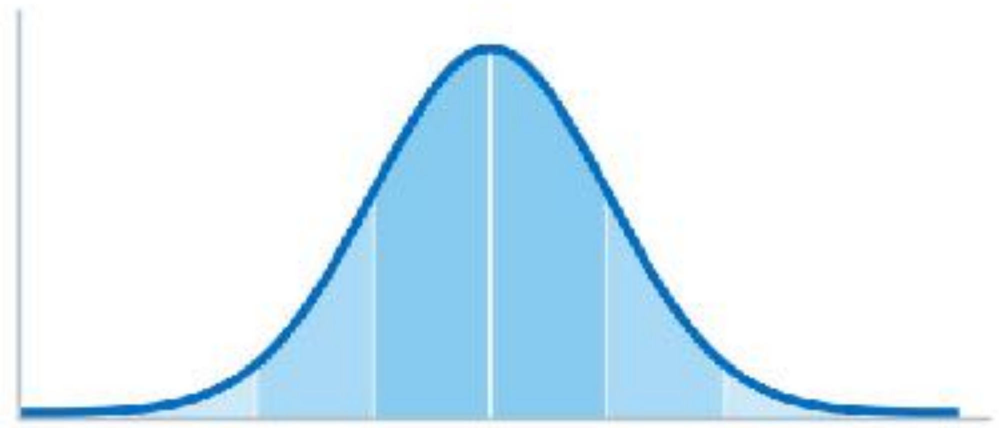

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Universal random variable **(0d index set i.e. point)**

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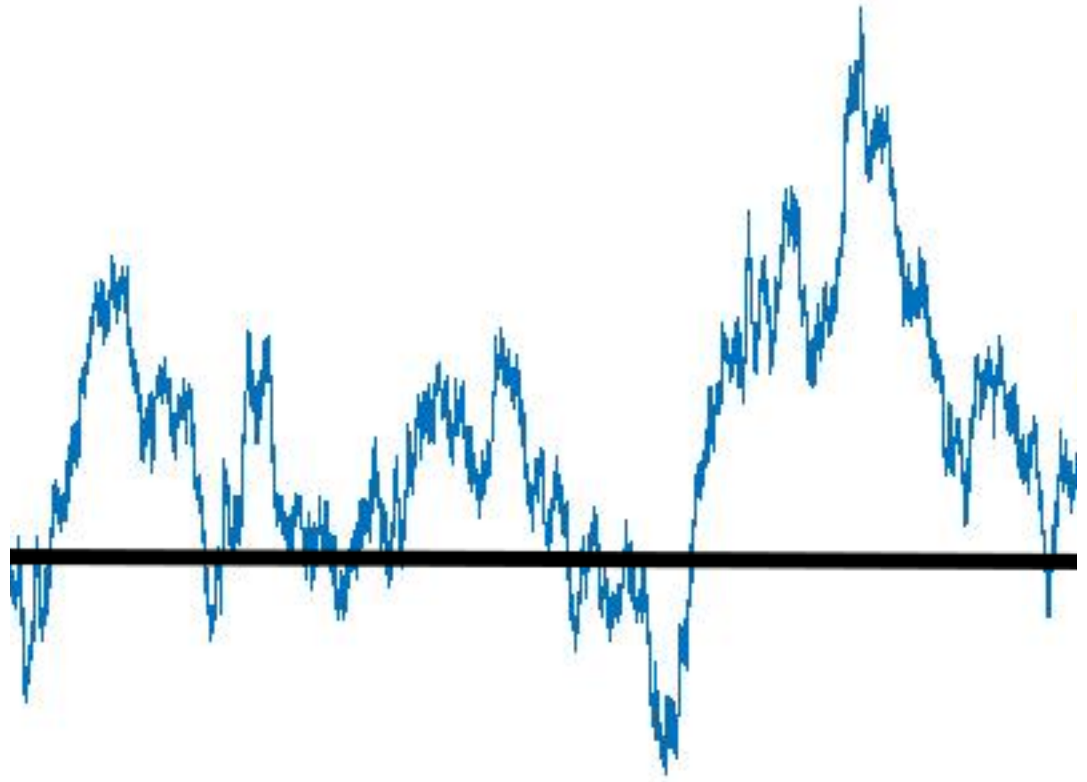

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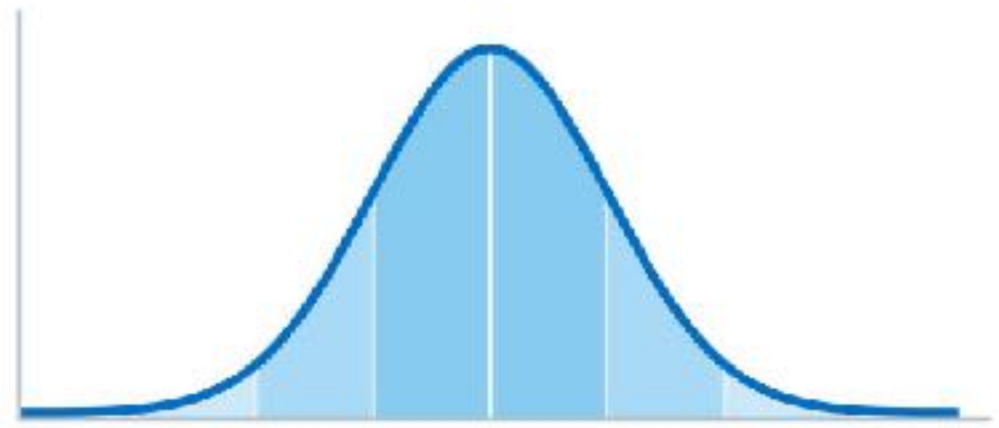

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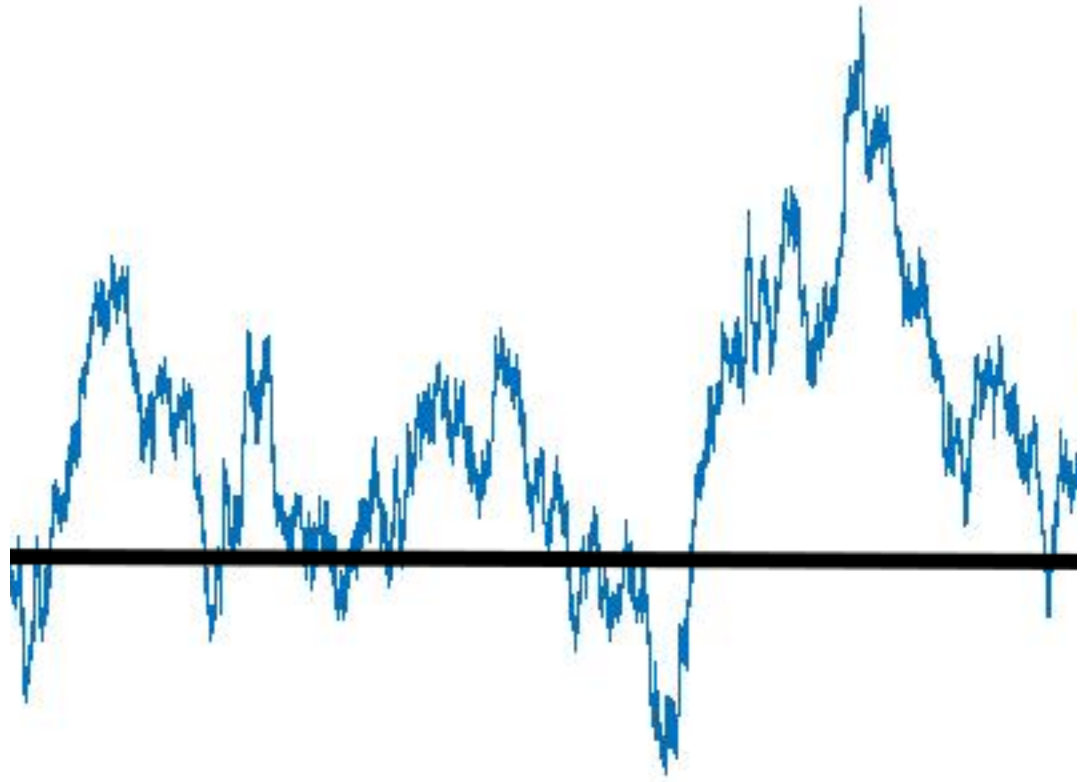

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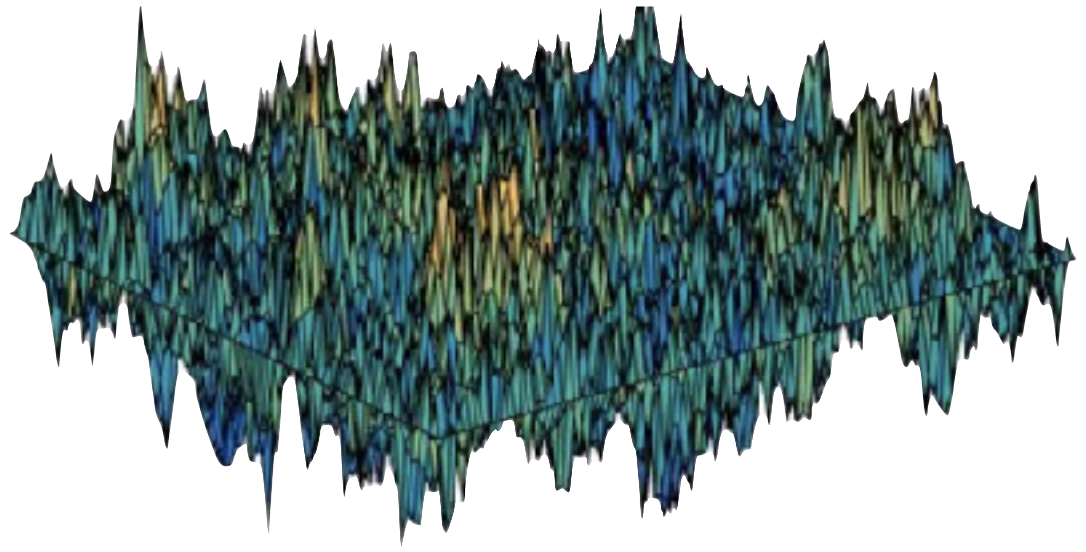


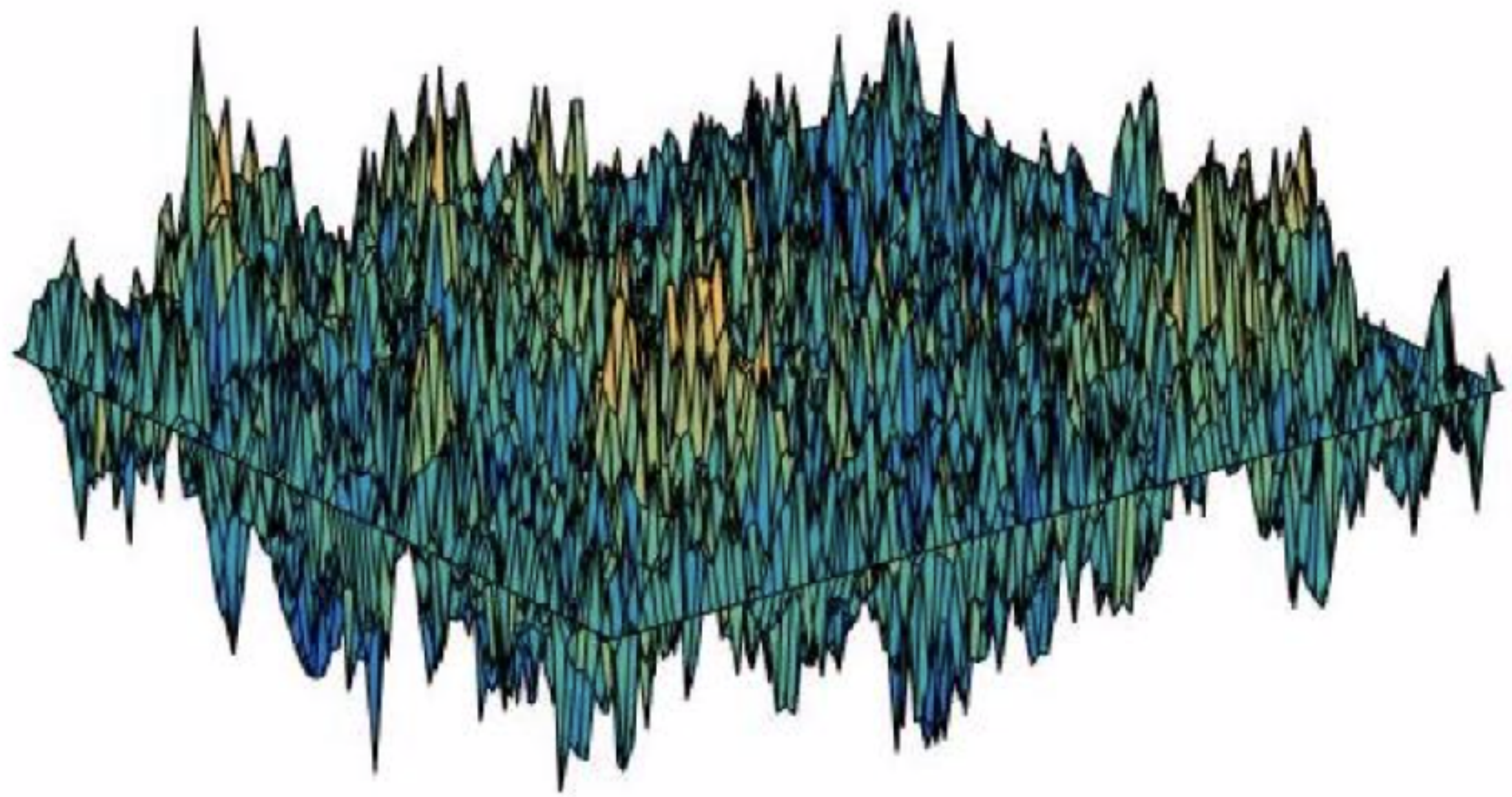
Universal random field: d**-diml index set?**

Gaussian free field: Definition

(With 0 boundary conditions, in the unit ball $\mathbb{B} \subset \mathbb{R}^d$, $d \geq 1$)

- **GFF** = "random function"

$$\text{"}\mathbb{P}(h \in df)\text{"} \propto \exp(-\frac{1}{2}\int_D |\nabla f|^2)$$



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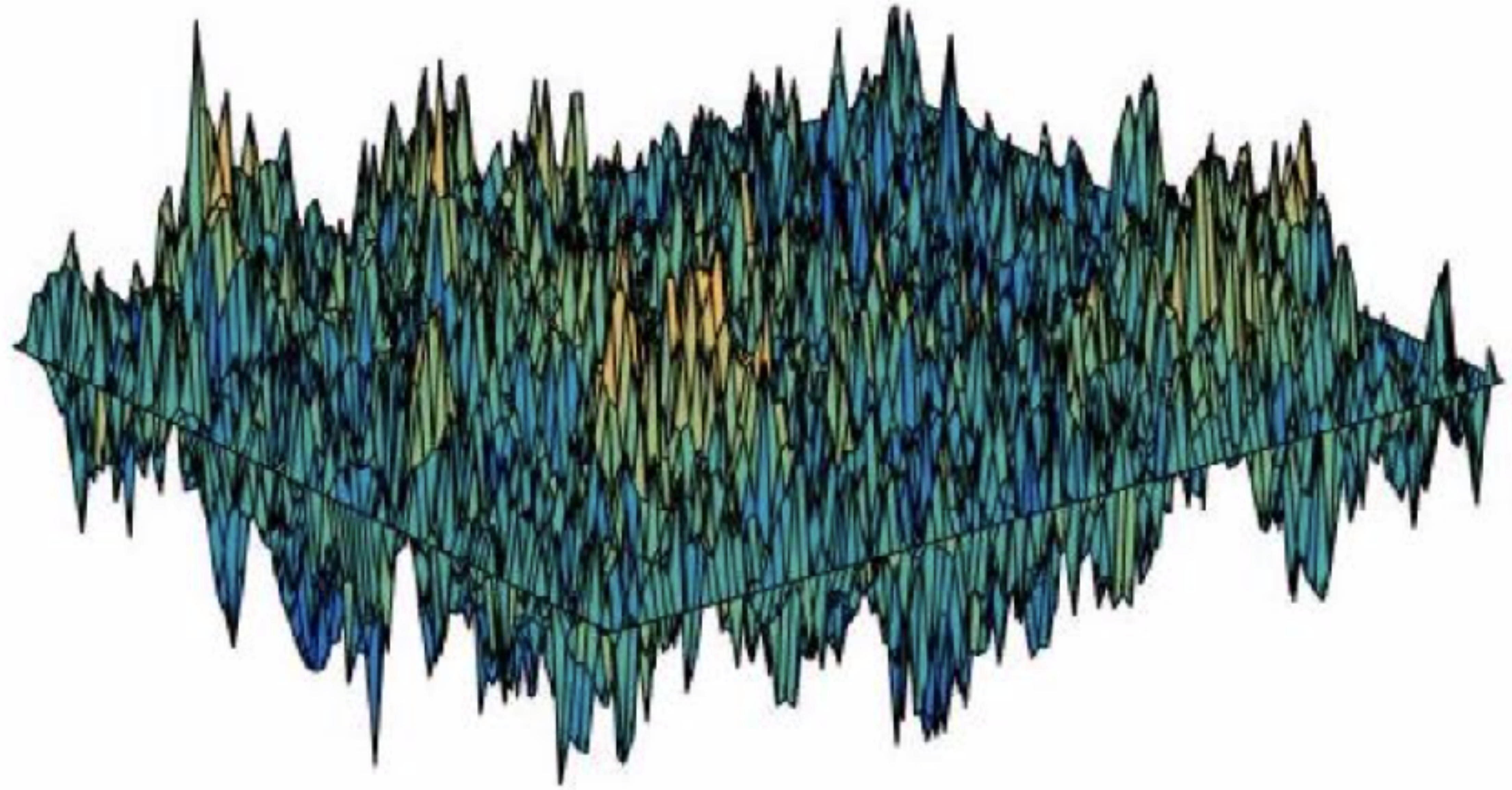

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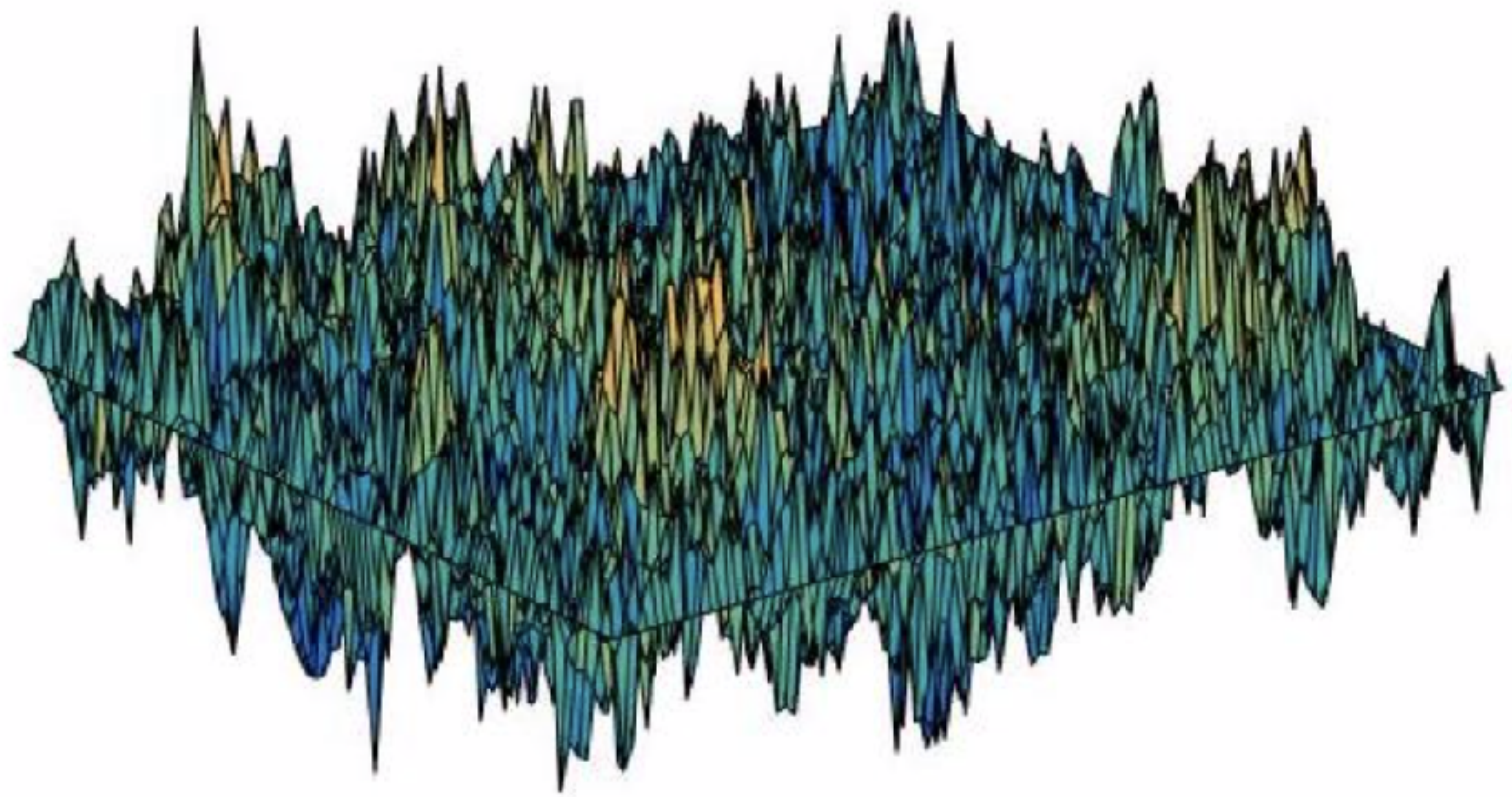

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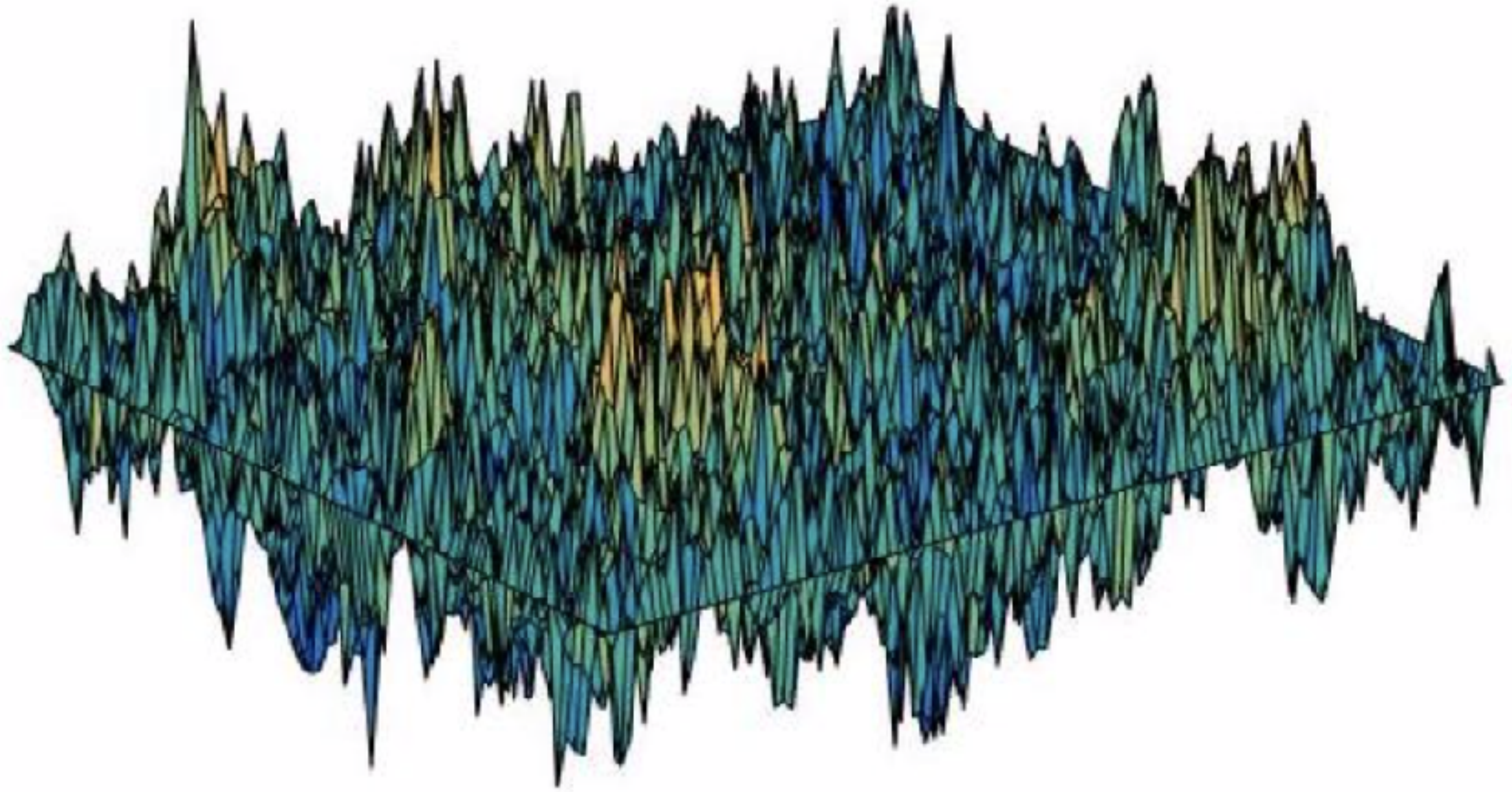

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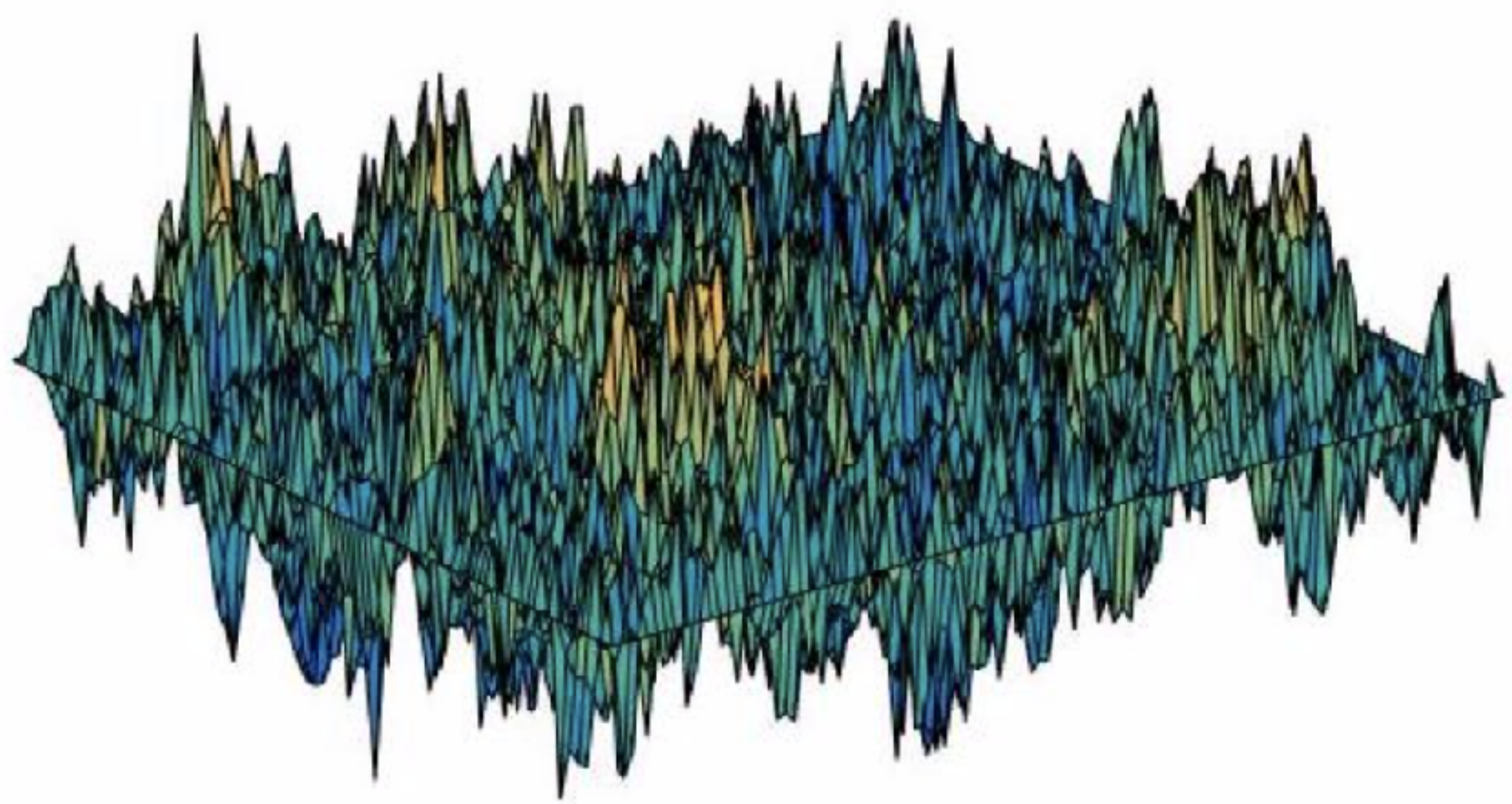

To rigorously define the GFF, need to let it live in the space of **random distributions**, or **generalised functions**

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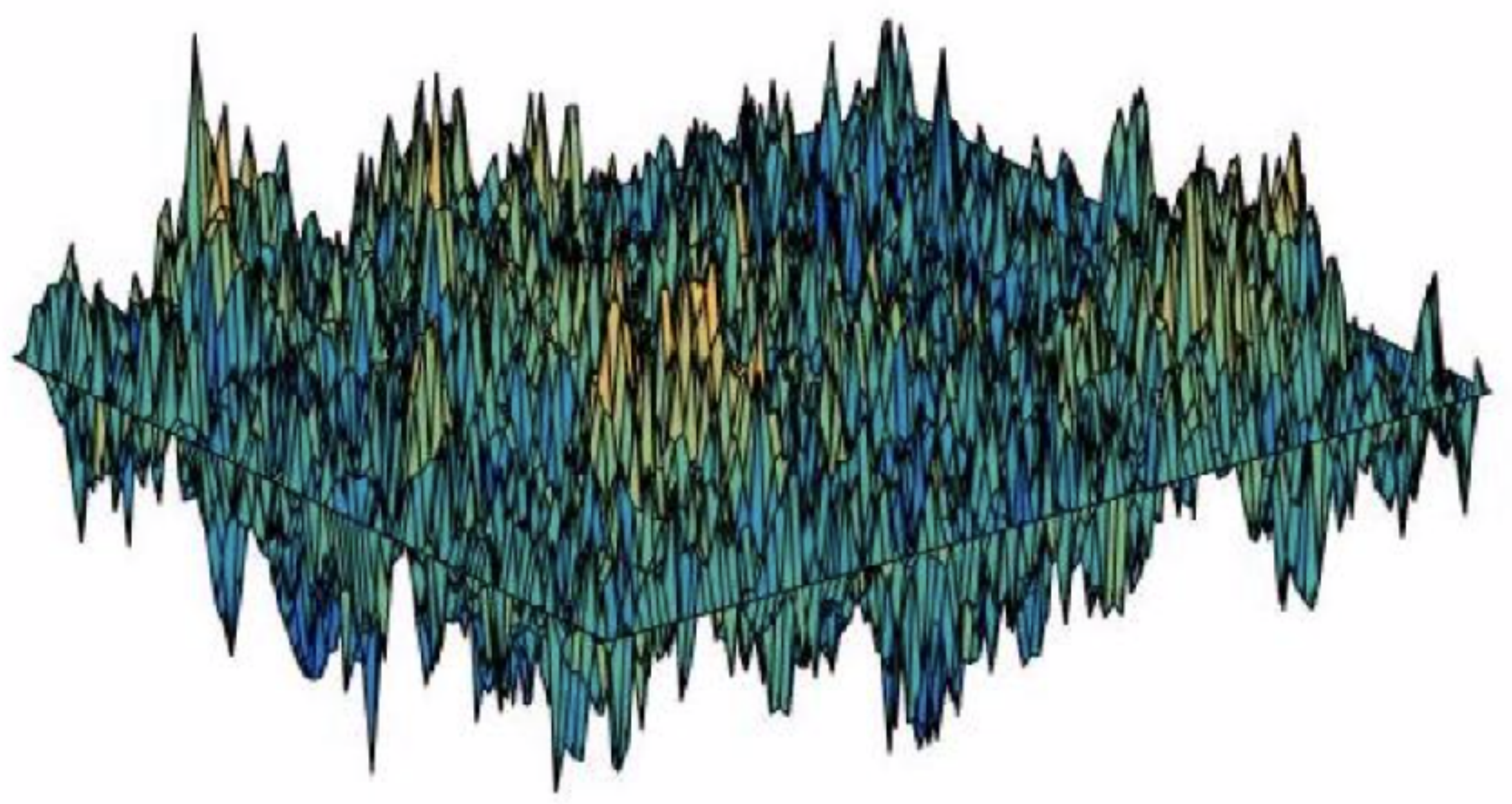
Random Schwartz distribution h such that $(h, f)_{f \in C_c^\infty(\mathbb{B})}$ is **centered, Gaussian** with

$$\mathbb{E}((h,f)(h,g)) = \iint_{\mathbb{B}^2} f(x) G^{\mathbb{B}}(x,y) g(y)\, dxdy$$

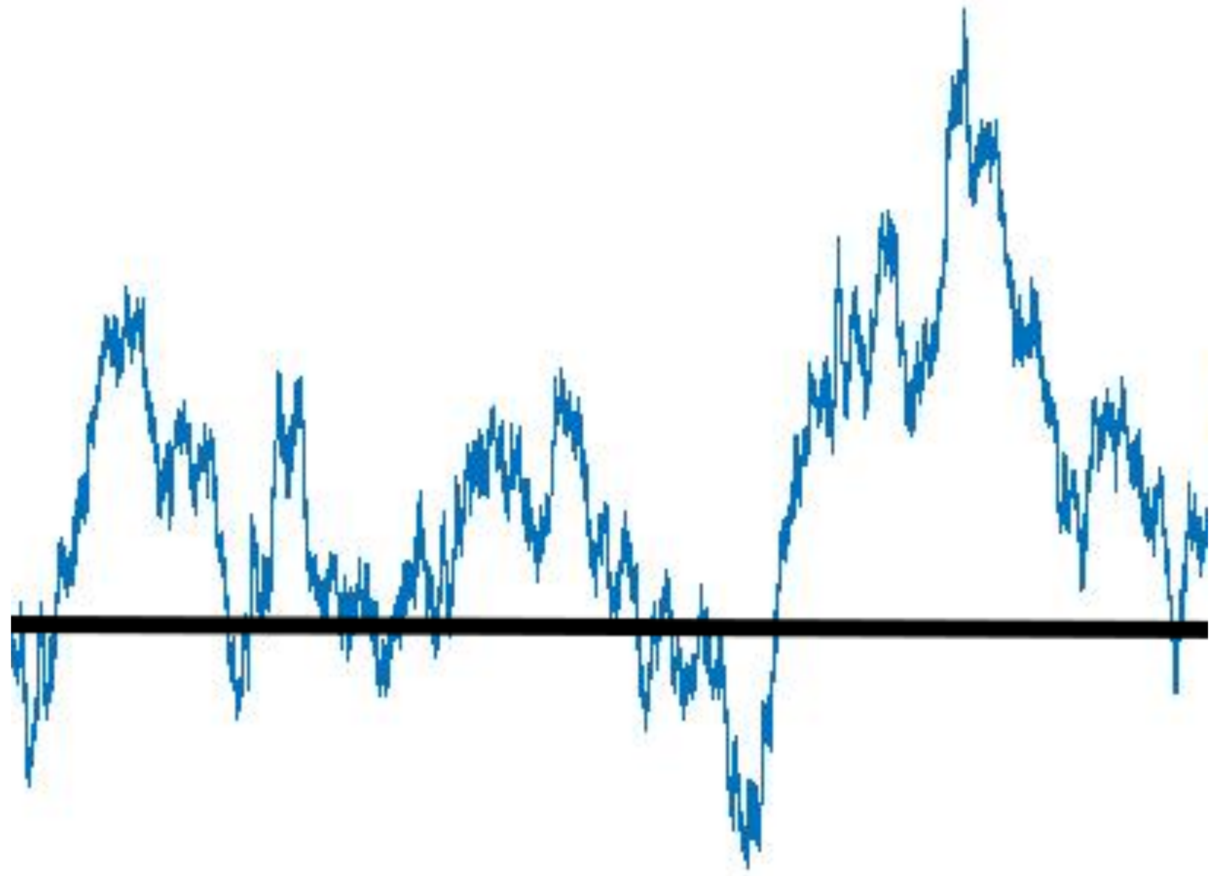
for all $f, g \in C_c^\infty(\mathbb{B})$

$G^{\mathbb{B}}$ is the Greens function for the Laplacian with zero boundary conditions in $\mathbb{B}$



Example: $d = 1$

Brownian bridge



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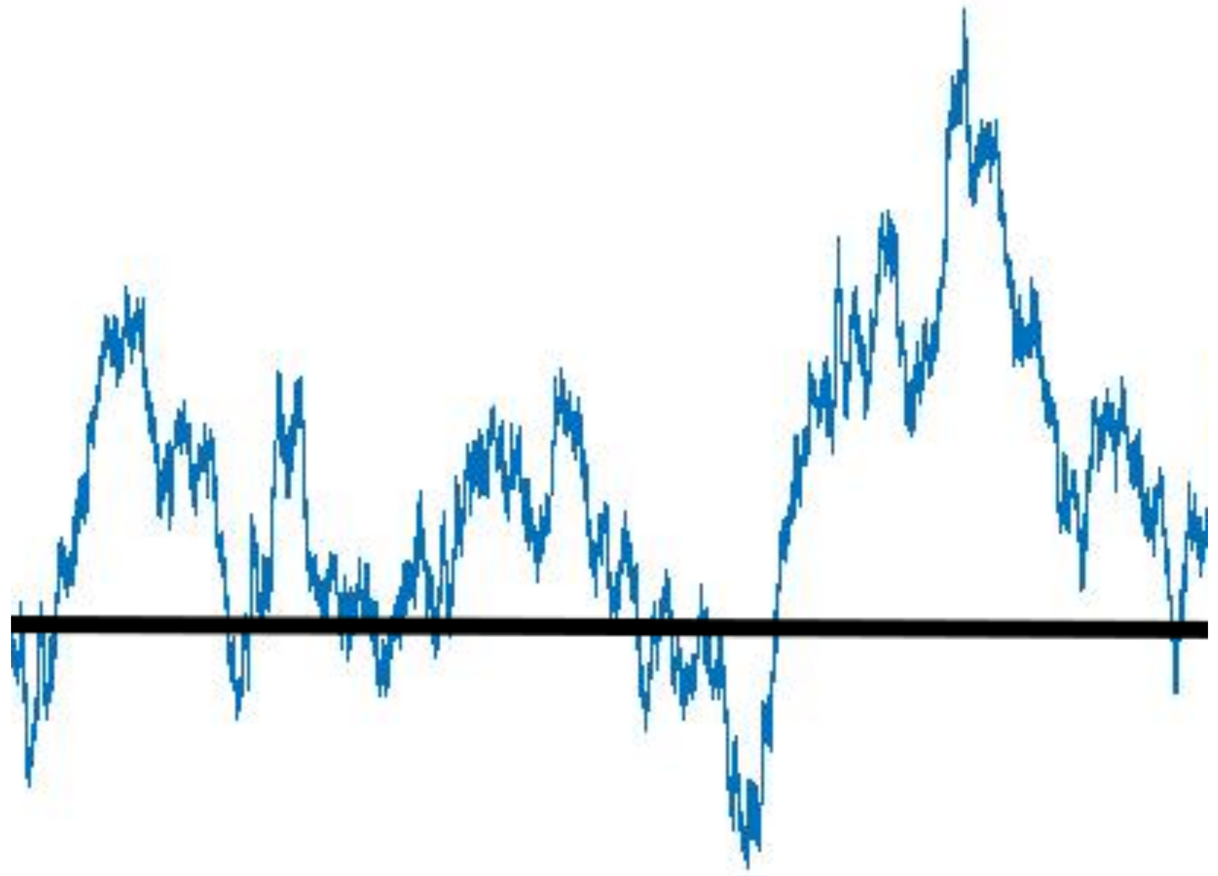

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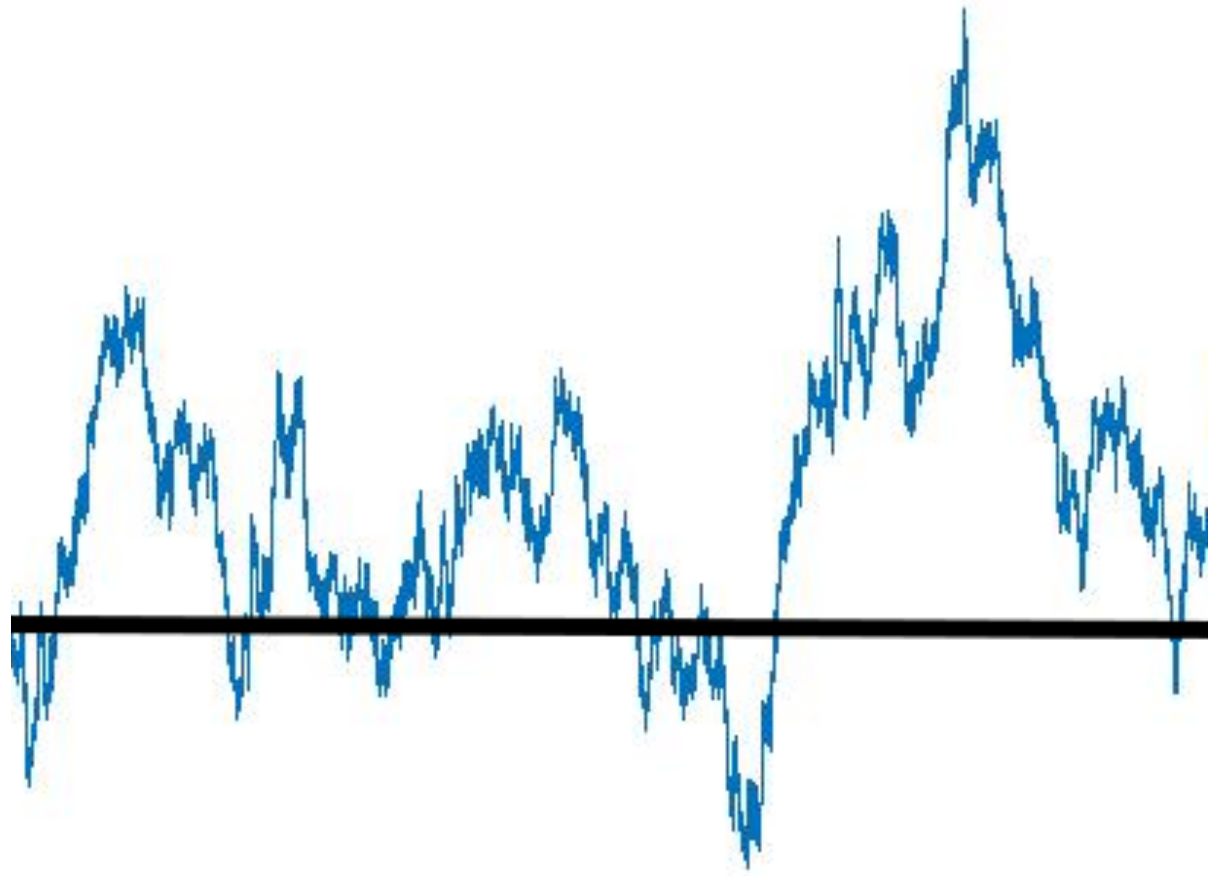

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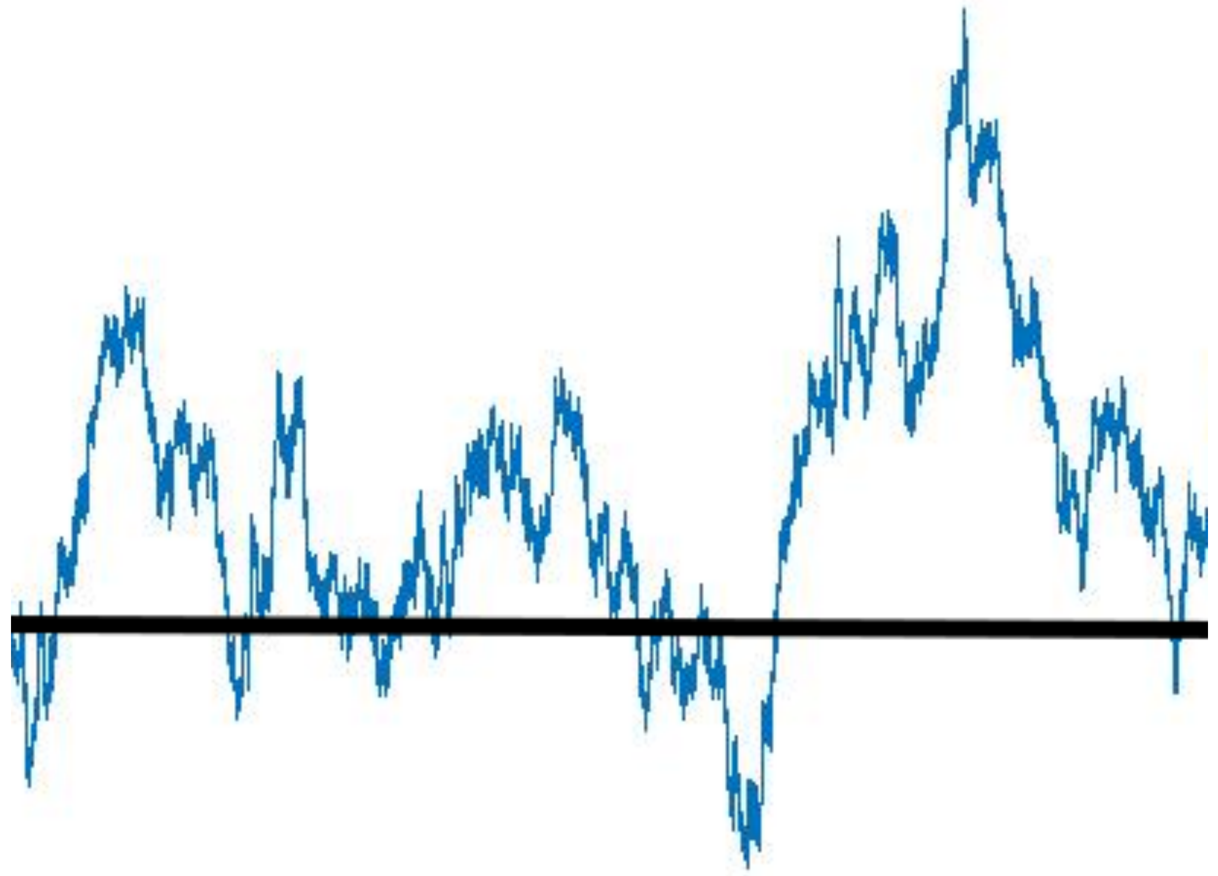

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- **Lots of characterisations** (at least for Brownian motion)

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Planar GFF

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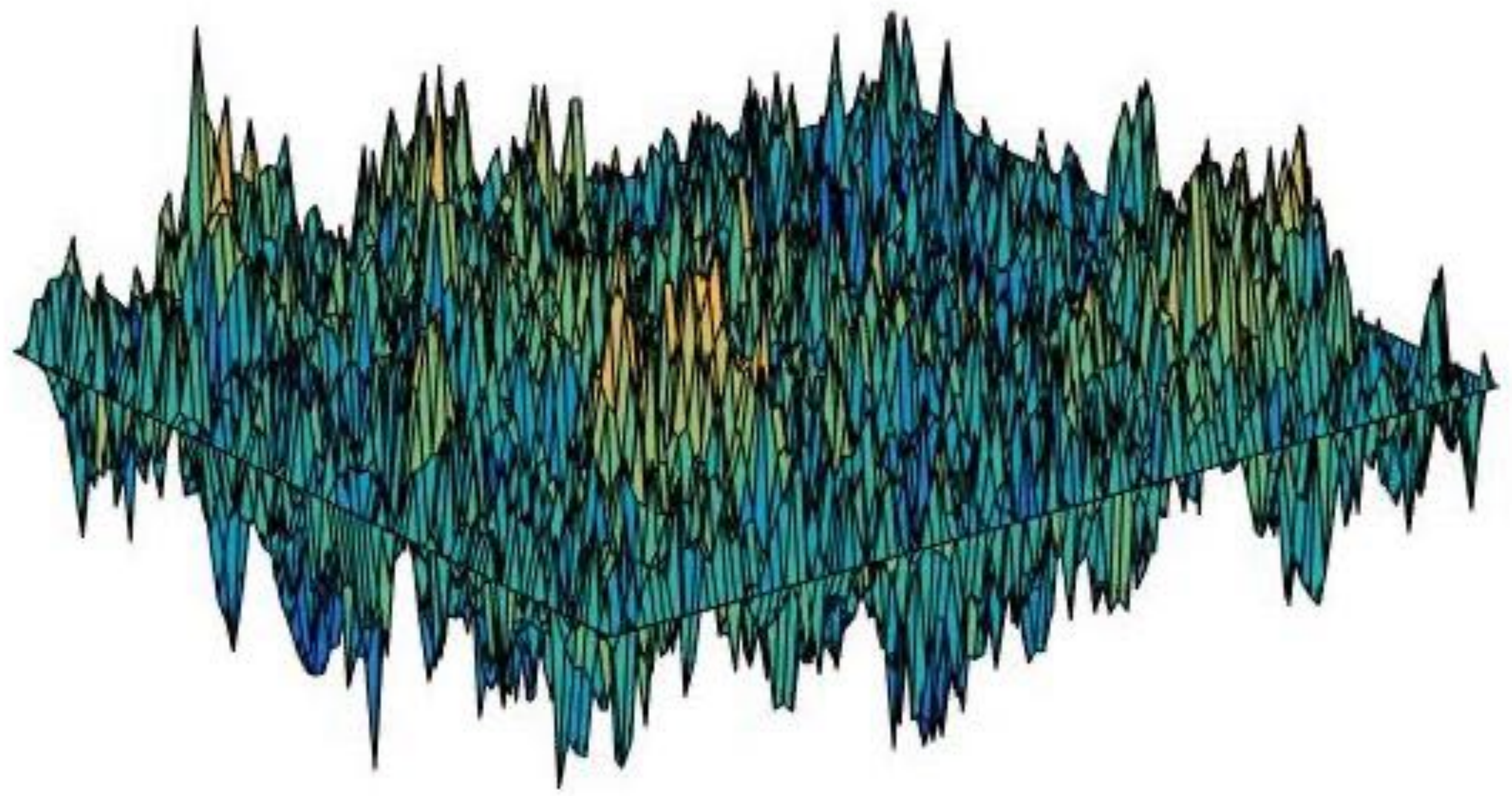

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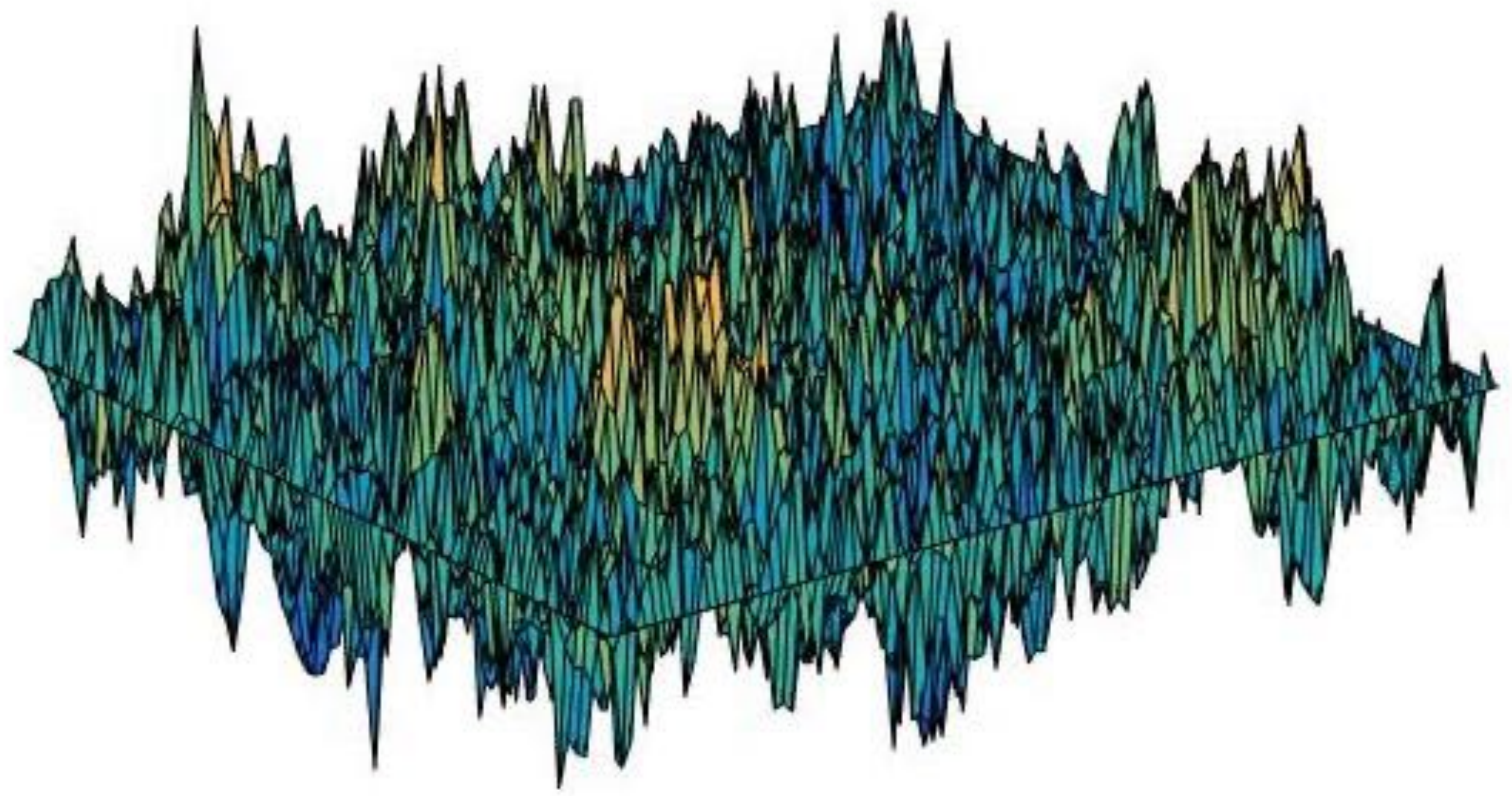

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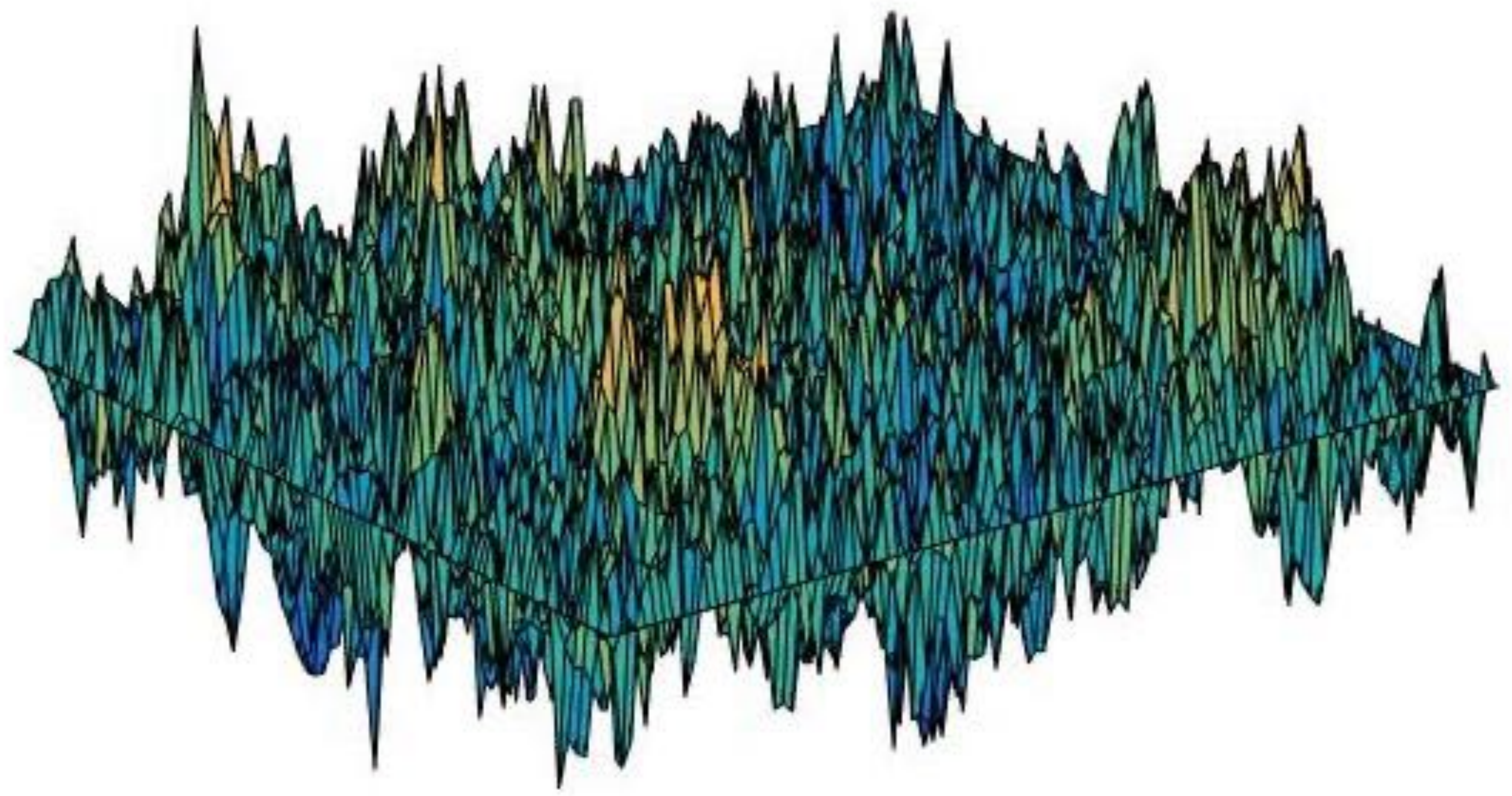

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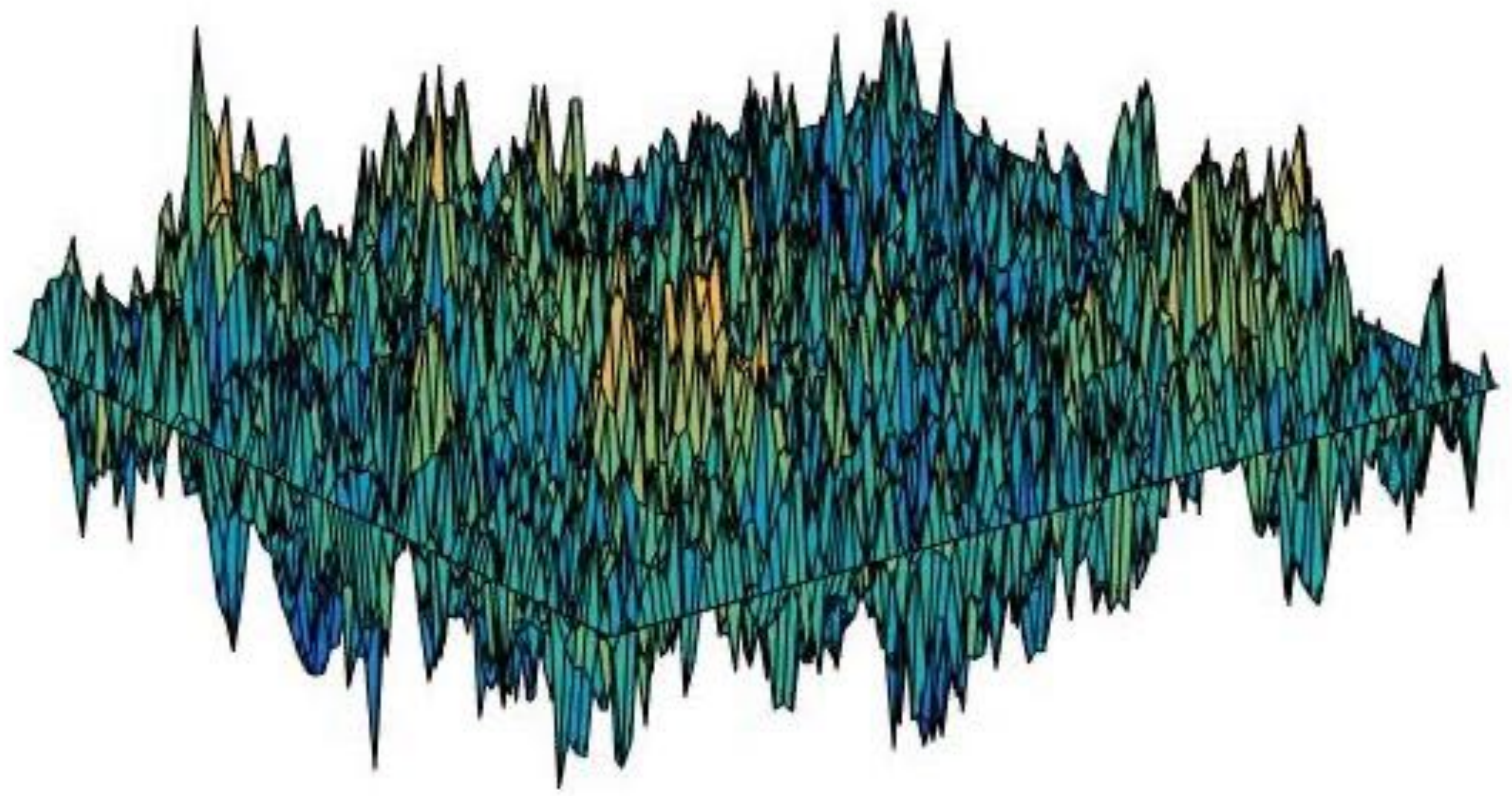

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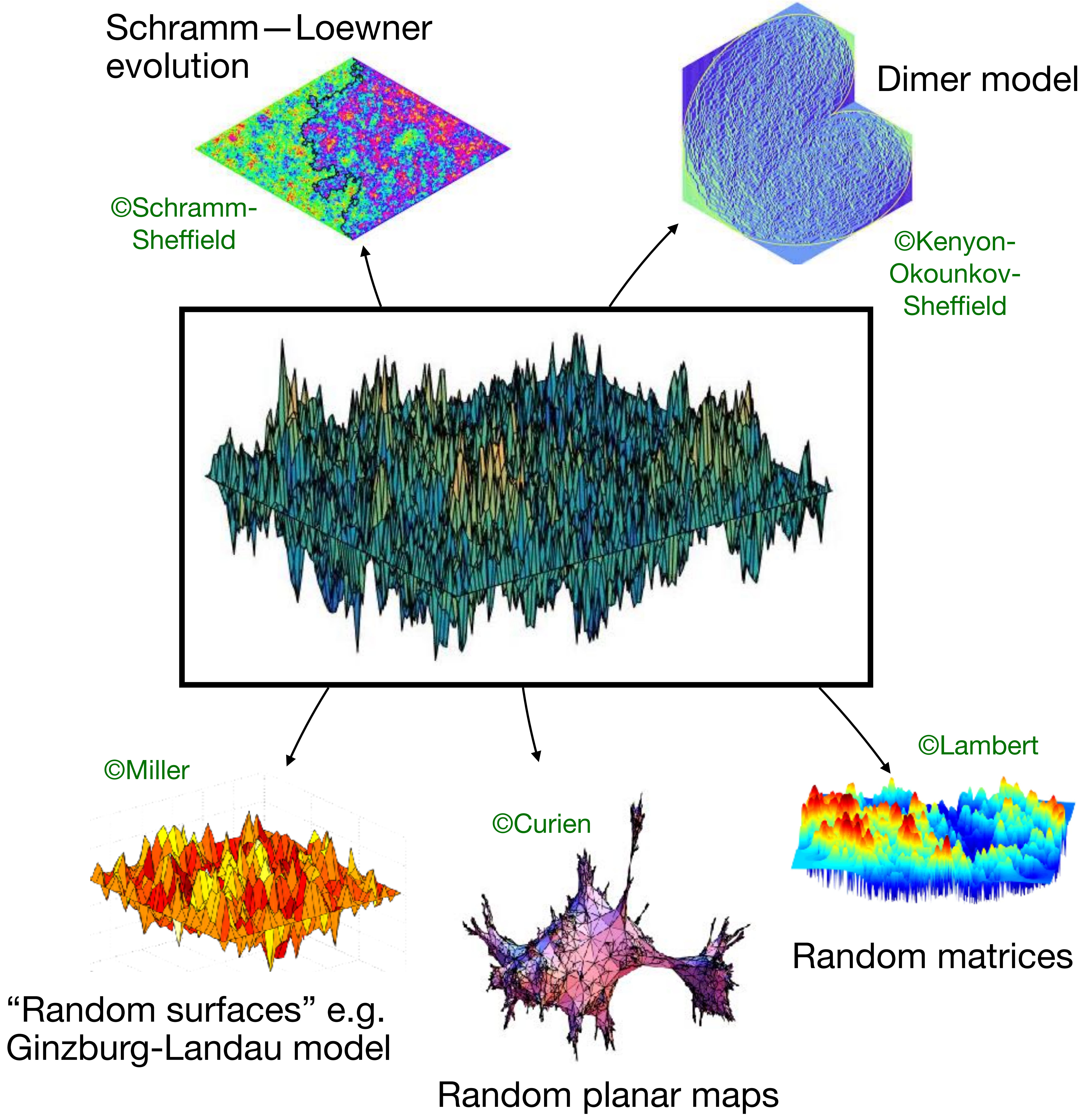
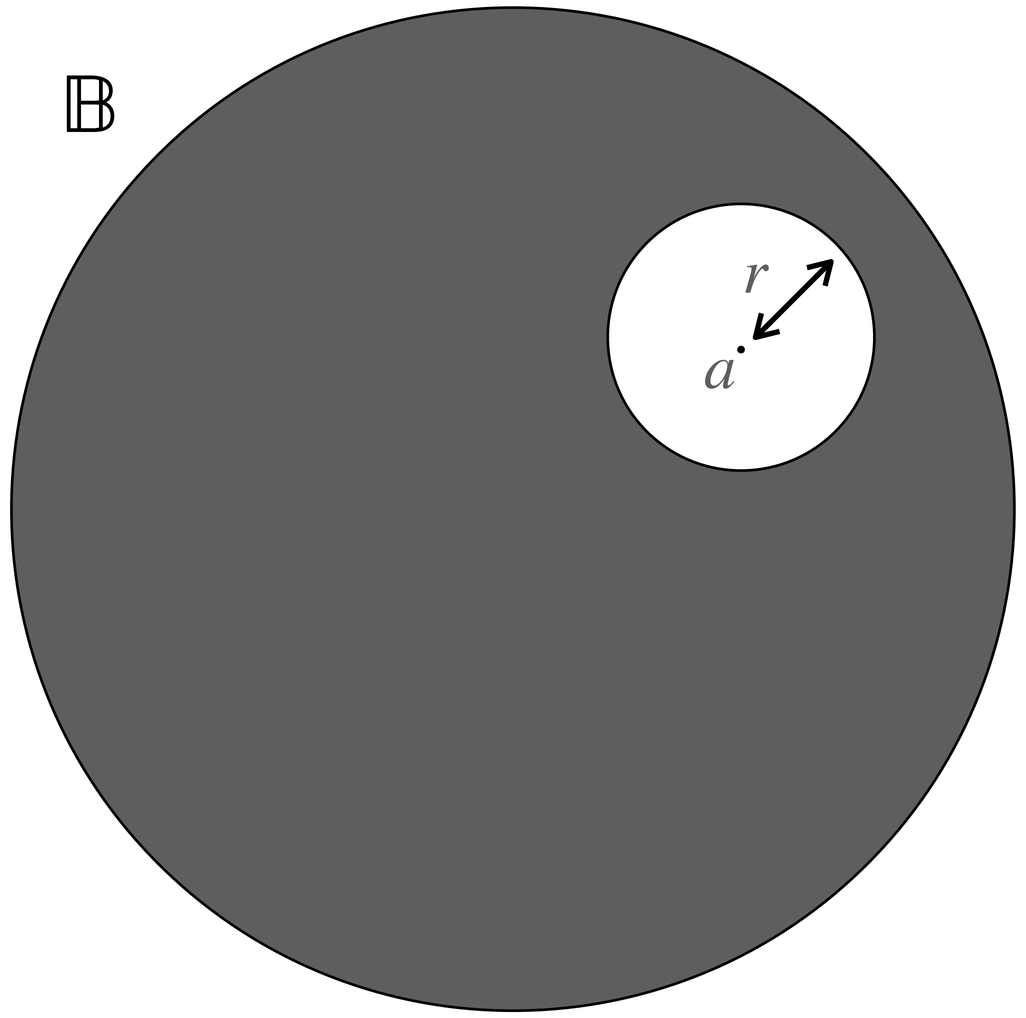


The domain Markov property



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Scaled copy of field + independent harmonic function

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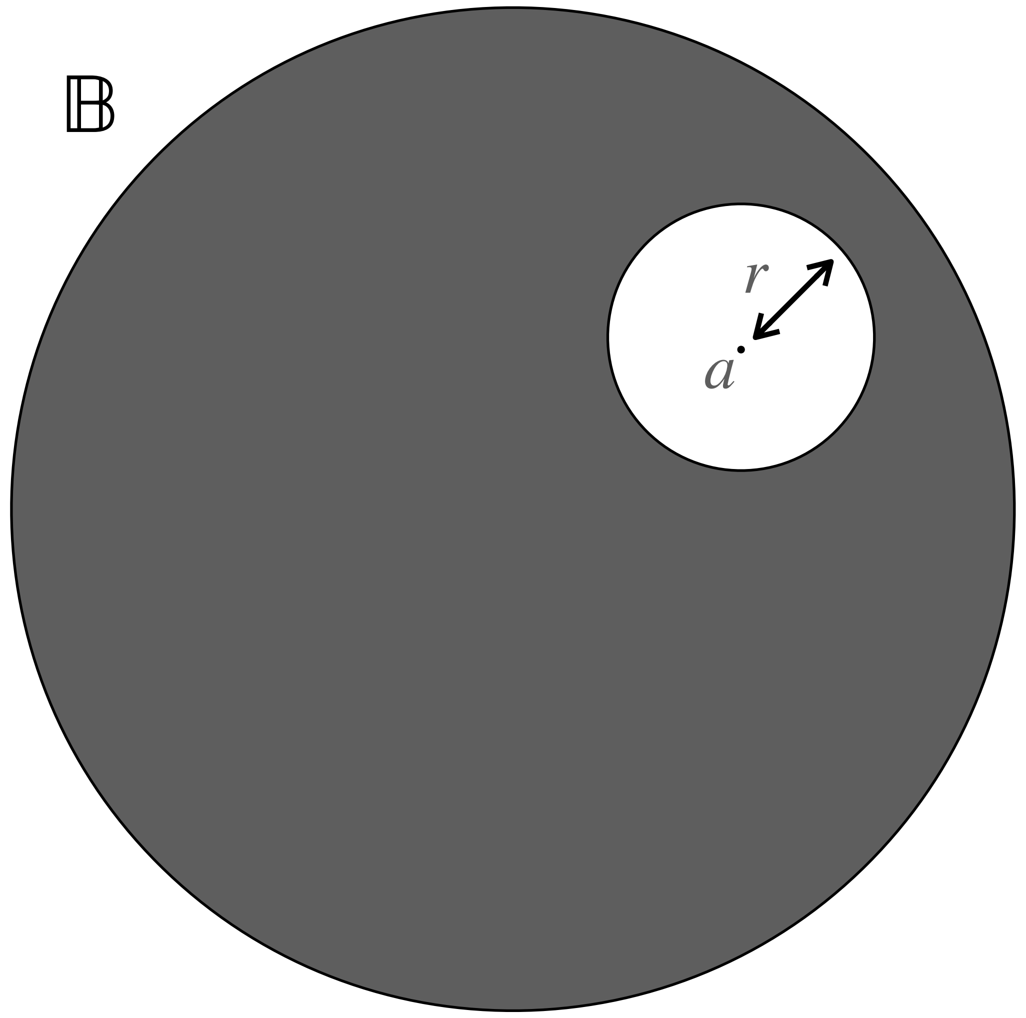

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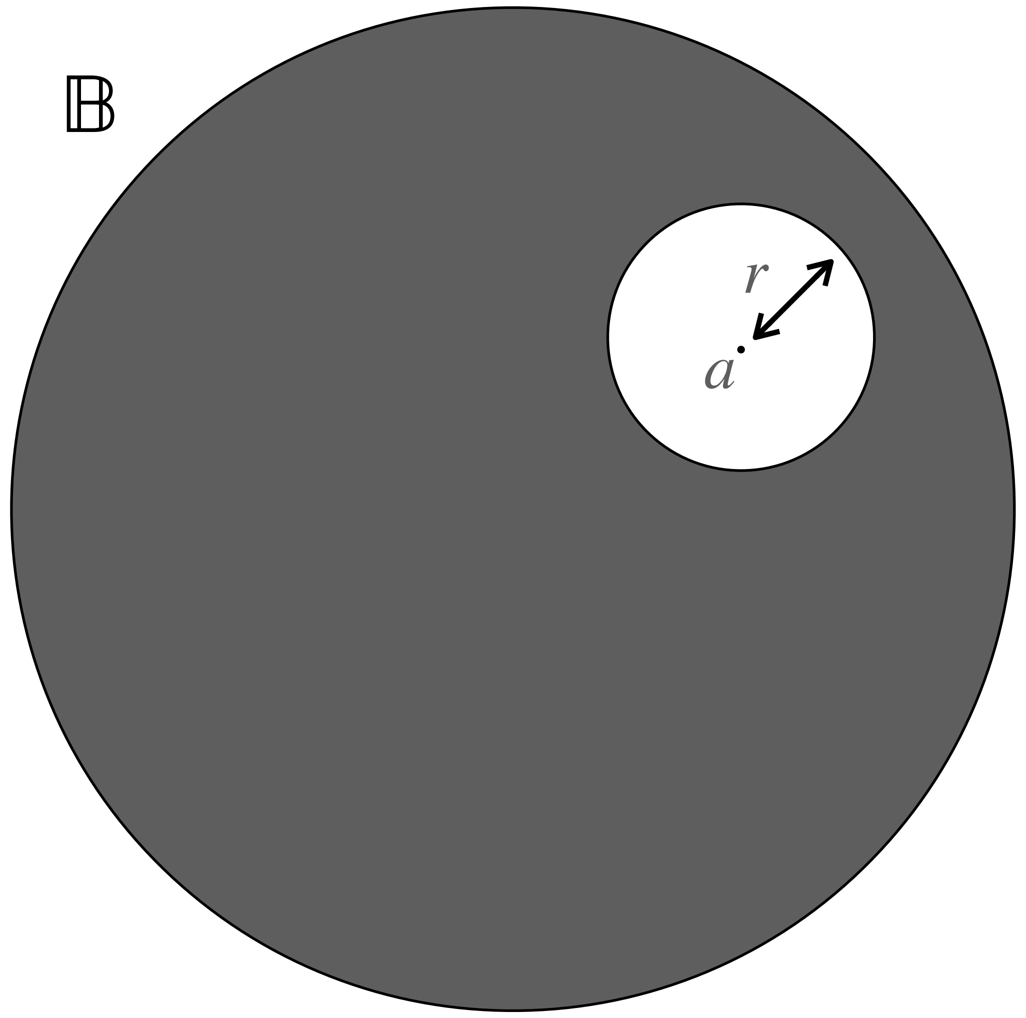

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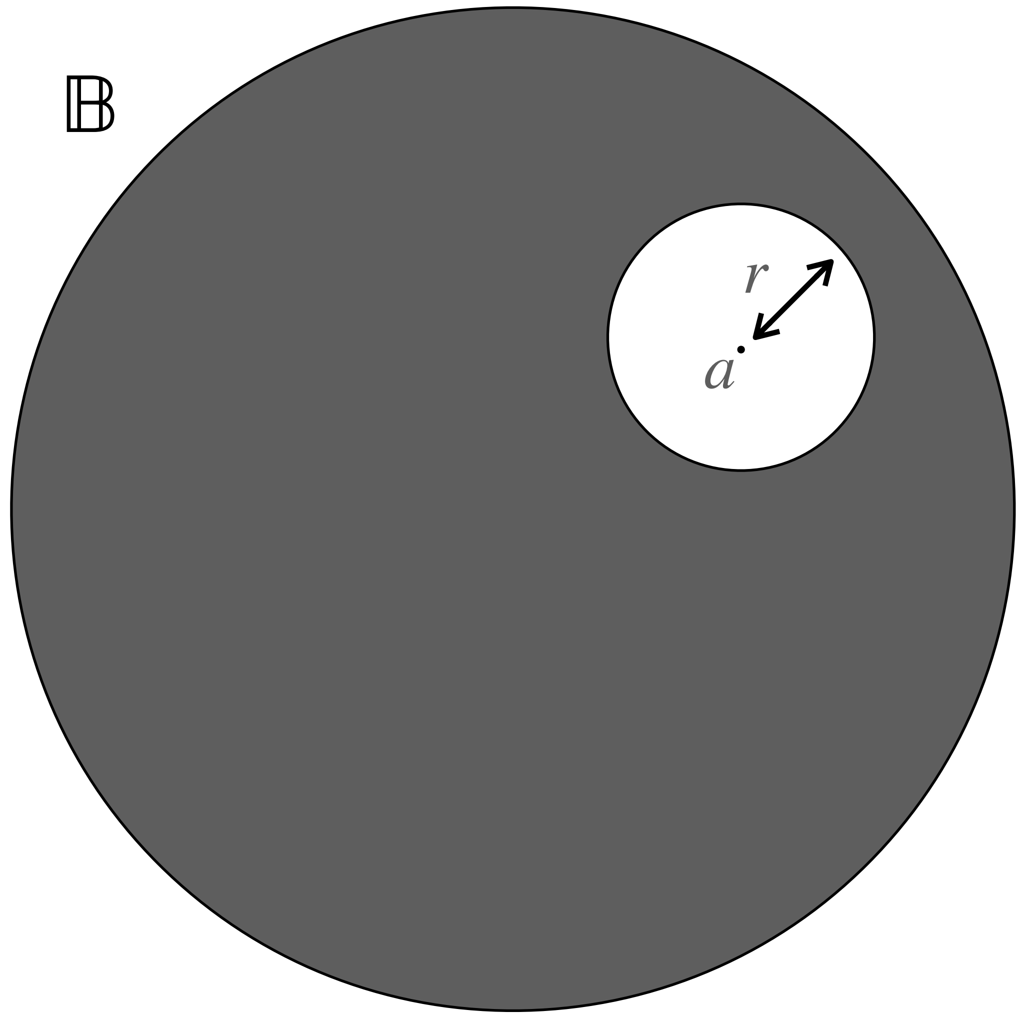

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- $h^{a+r\mathbb{B}}$ and $\varphi^{a+r\mathbb{B}}$ are **independent**

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$d \geq 2$, $\mathbb{B} \subset \mathbb{R}^d$ unit ball

Aru-P. '21

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$(h, f_n) \to 0$ in L^2 for $(f_n)_{n \geq 0}$ smooth & positive, with support $\to \partial \mathbb{B}$ and $\sup_n \Big(\sup_{r>1} \sup_{x,y \in \partial r\mathbb{B}} | f_n(x)/f_n(y) | + \| f_n \|_{L^1(\mathbb{B})} \Big) < \infty)$

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- **Harmonicity** in the Markov property is **key**
- Are there other interesting fields characterised by a different Markov property? (e.g., stable bridges/fields, CLE nesting field?)
- What about **GFFs on other manifolds?**

Idea for the proof

Outline

Two steps

- Covariance is the Greens’ function (simpler step)
- Gaussianity (more challenging)

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Idea of proof

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Key ingredient: Suppose that for $y \in \mathbb{B}$, $k_y(x)$ is a **harmonic** function defined in $\mathbb{B}\backslash\{y\}$, such that $k_y(x) - bs(|x-y|)$ is bounded in a neighbourhood of y for some $b > 0$ and such that $(k_y, f_n)_{L^2} \to 0$ for any sequence of functions f_n as in our zero boundary condition. Then $k_y(x) = bG^{\mathbb{B}}(x, y)$ for all $x \neq y$; $x, y \in \mathbb{B}$.

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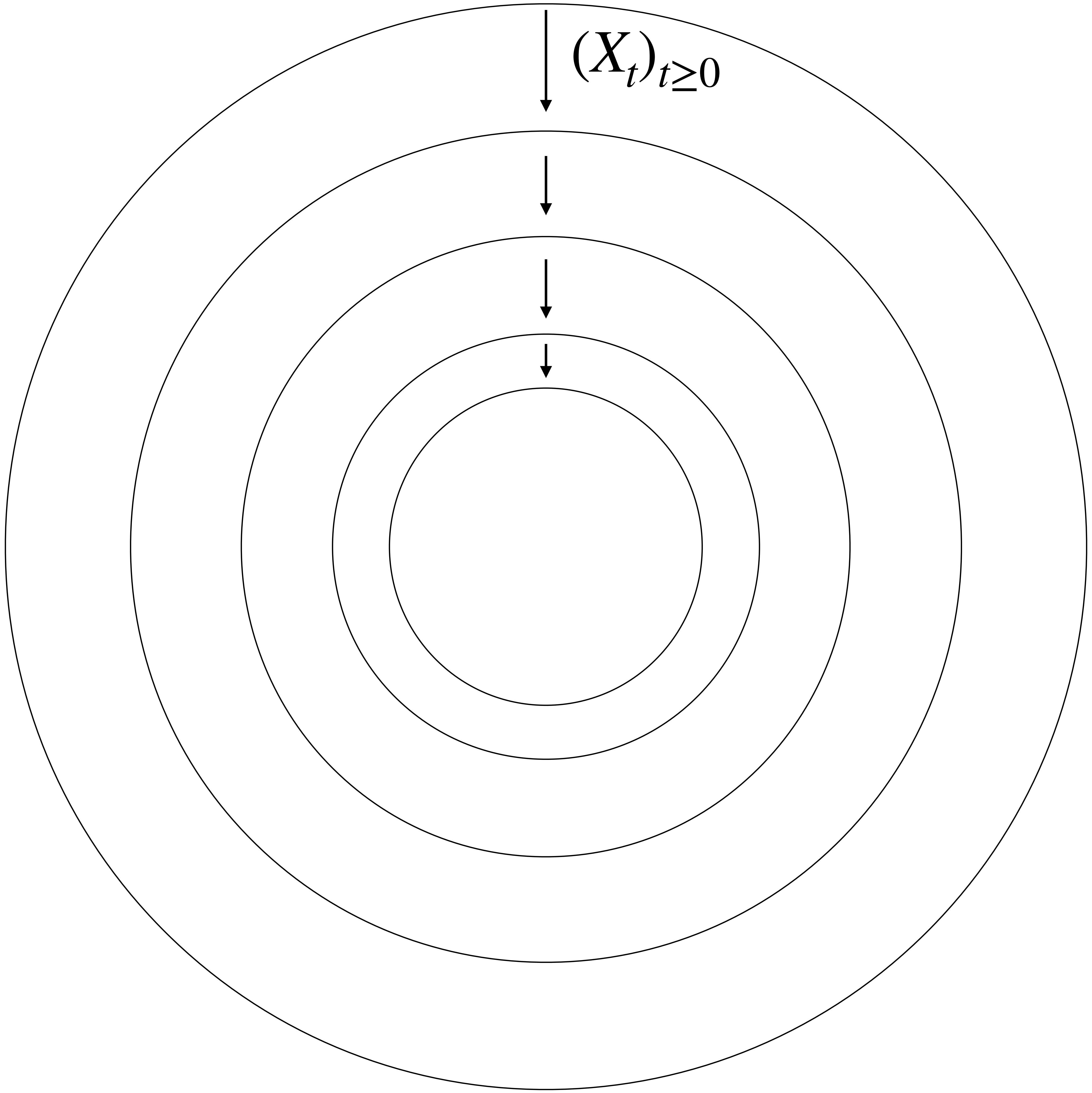
Harmonicity + scaling + boundary conditions $\Rightarrow$ Greens' function

This condition can be checked quite easily using the assumptions (esp. DMP)

Gaussianity

Warm up

$$X_t := \oint_{|x|=e^{-t}} \text{''}h(x)\,dx\text{''}$$



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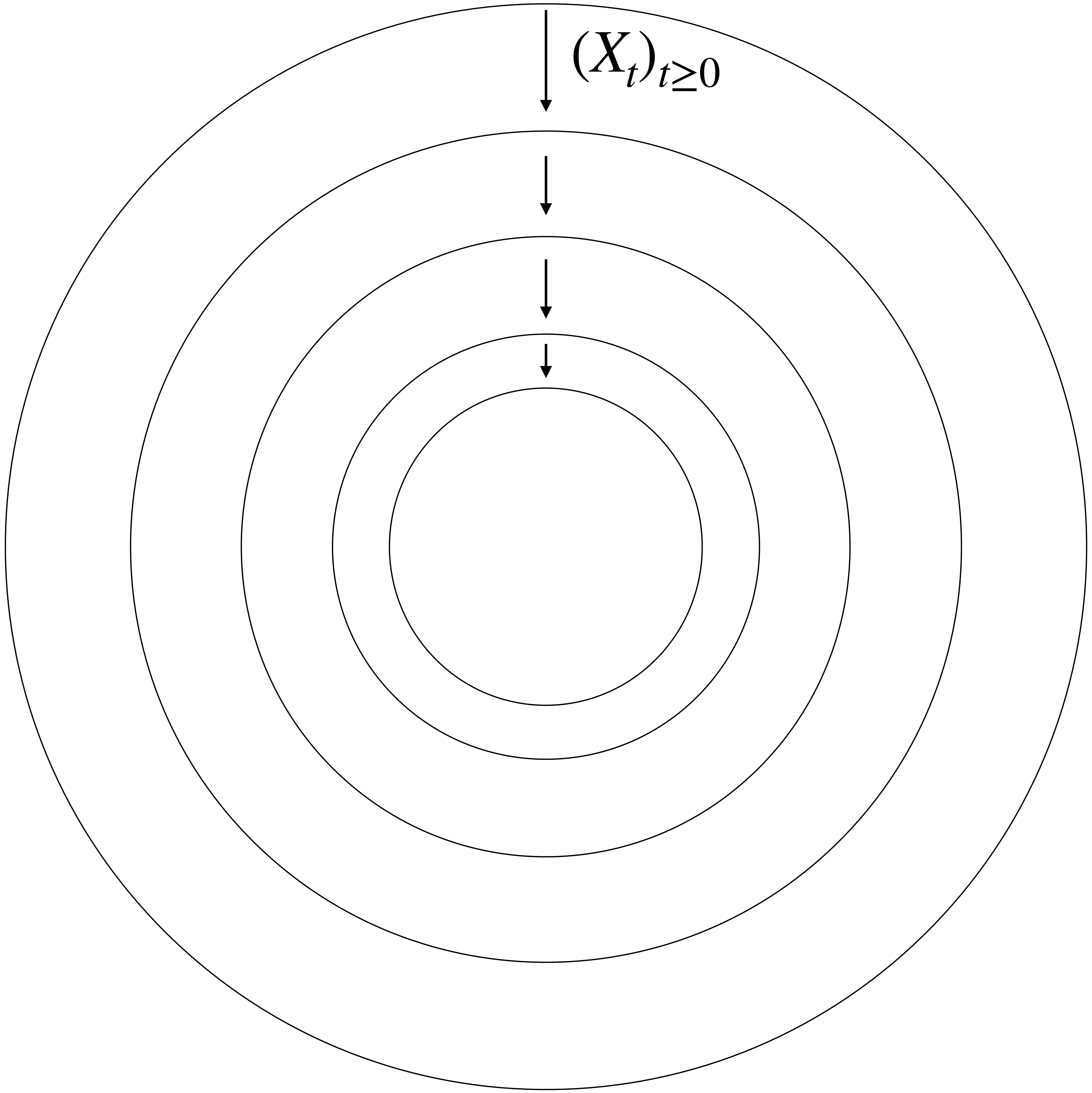

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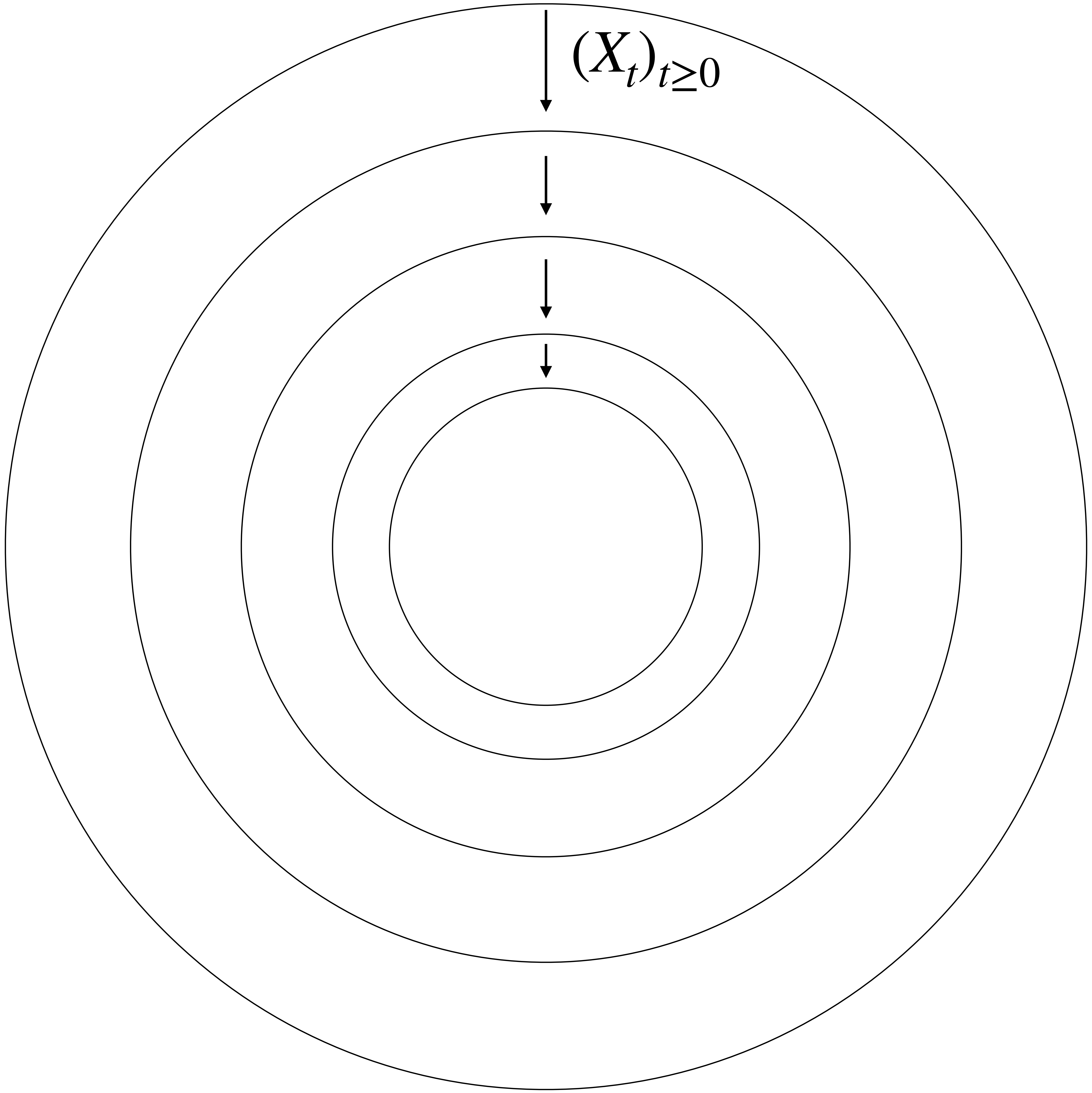
- Using the 4th moment assumption, and Kolmogorov's criterion, also has a **continuous modification**



X_t is the "average value" of h on the spherical shell of radius e^{-t}

Gaussianity

Warm up

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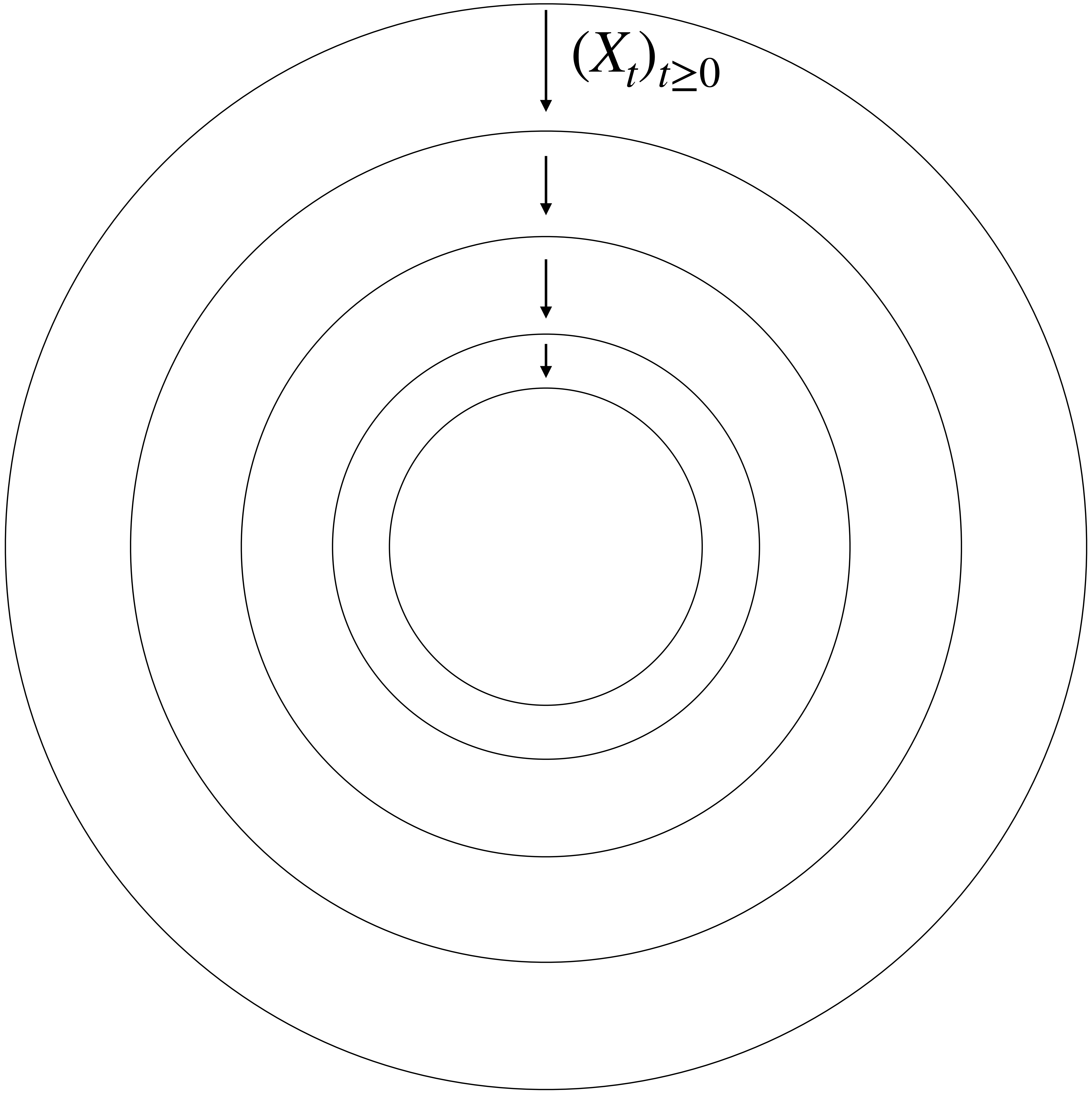

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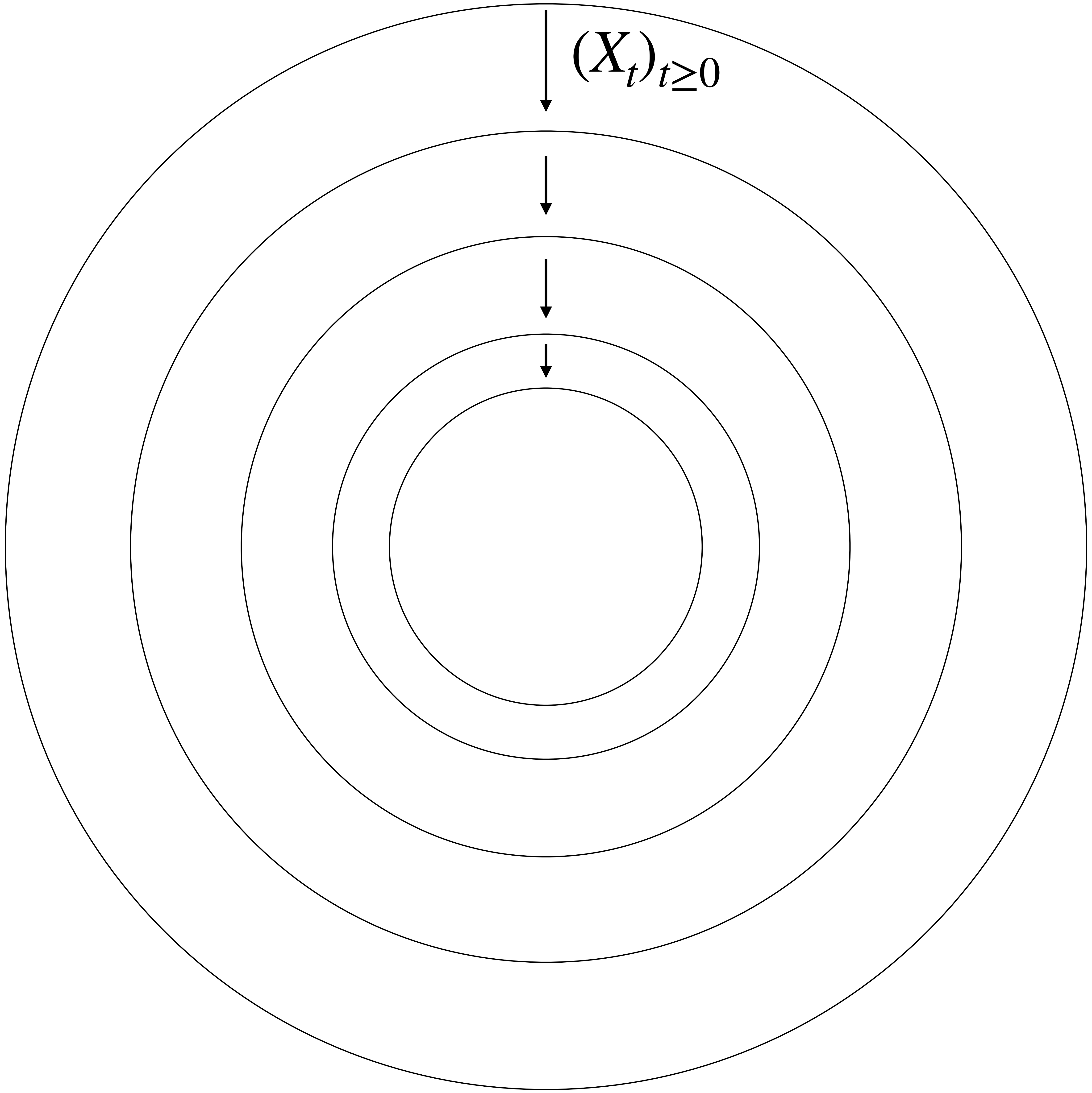
- In $d \geq 3$ everything is the same except the stationarity. Still get Gaussianity!



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Gaussianity

Spherical harmonics

Example In 2d,
$\psi_{n,1} = \sin(n\theta), \psi_{n,2} = \cos(n\theta)$ for
$n \geq 1$

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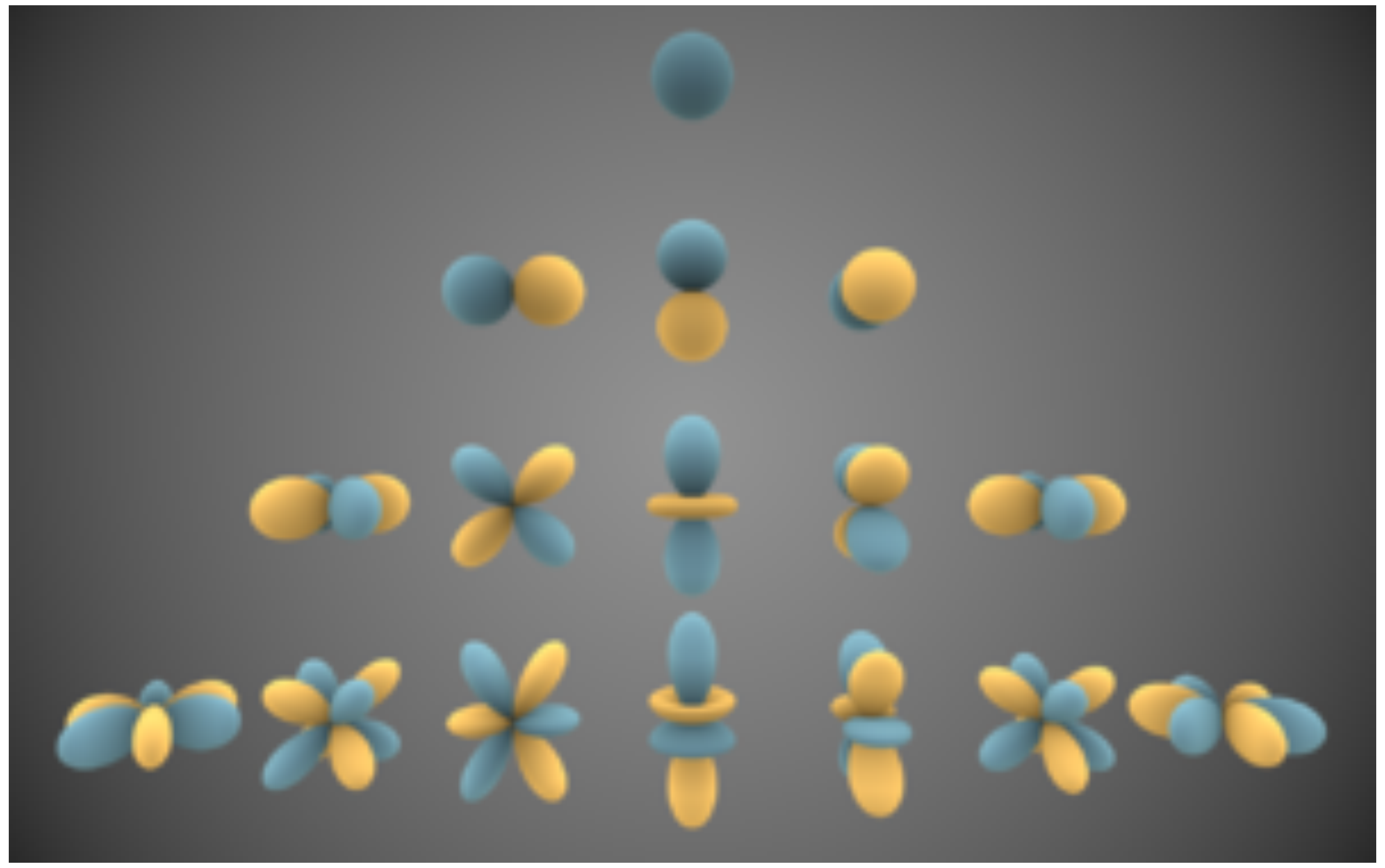

Let $(\psi_{n,j})_{n\geq 0, 1\leq j\leq M_n}$ be an orthonormal basis of **spherical harmonics** for $L^2(\partial\mathbb{B})$. In particular, $x \mapsto |x|^n \psi_{n,j}(x/|x|)$ is harmonic in $\mathbb{B}$

Gaussianity

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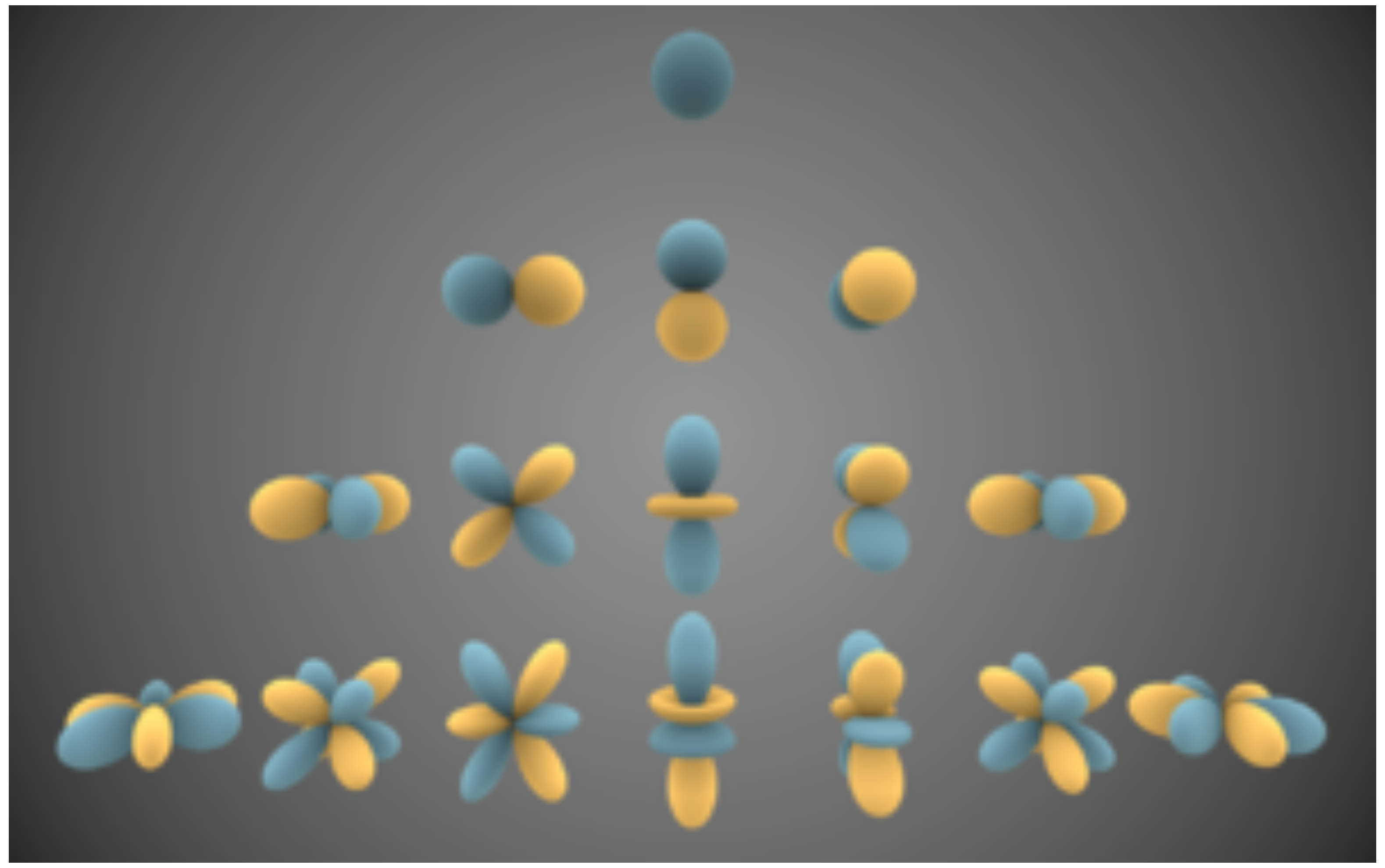

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- The same argument as for the spherical average case then gives that

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for $r \in (0,1]$ is a Gaussian process

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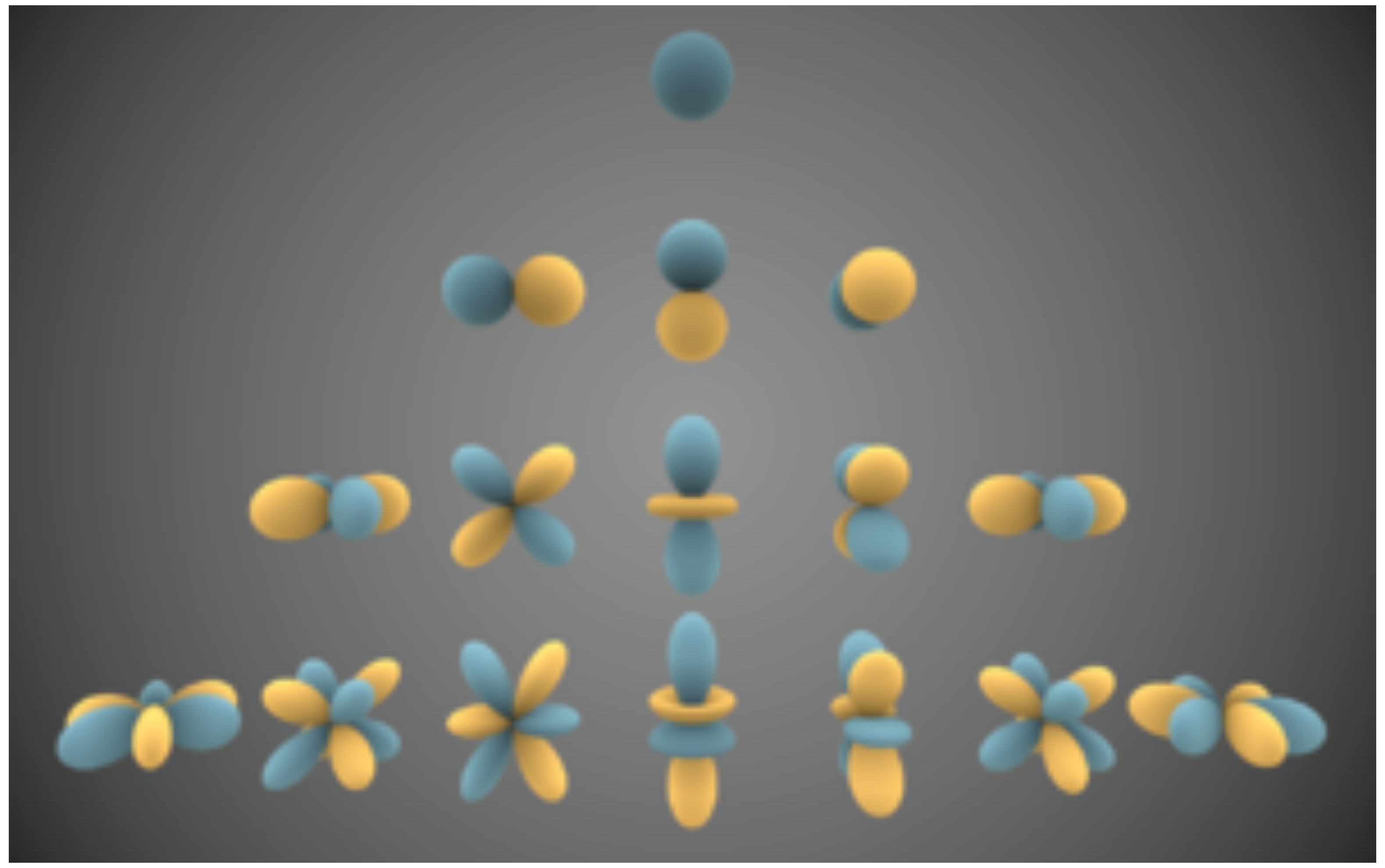

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for $r \in (0,1]$ is a Gaussian process

- Using Markov property again ⇒ "spherical harmonic averages", as a process indexed by the radius and the choice of harmonic $\psi_{n,j}$, is **Gaussian**

Gaussianity

Conclusion

Gaussianity

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- There exist radial functions $(f_{n,i})_{i,n\geq 0}$ such that

$$x \mapsto f_{n,i}(\,|x|\,)\,\psi_{n,j}(\frac{x}{|x|})$$

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Gaussianity

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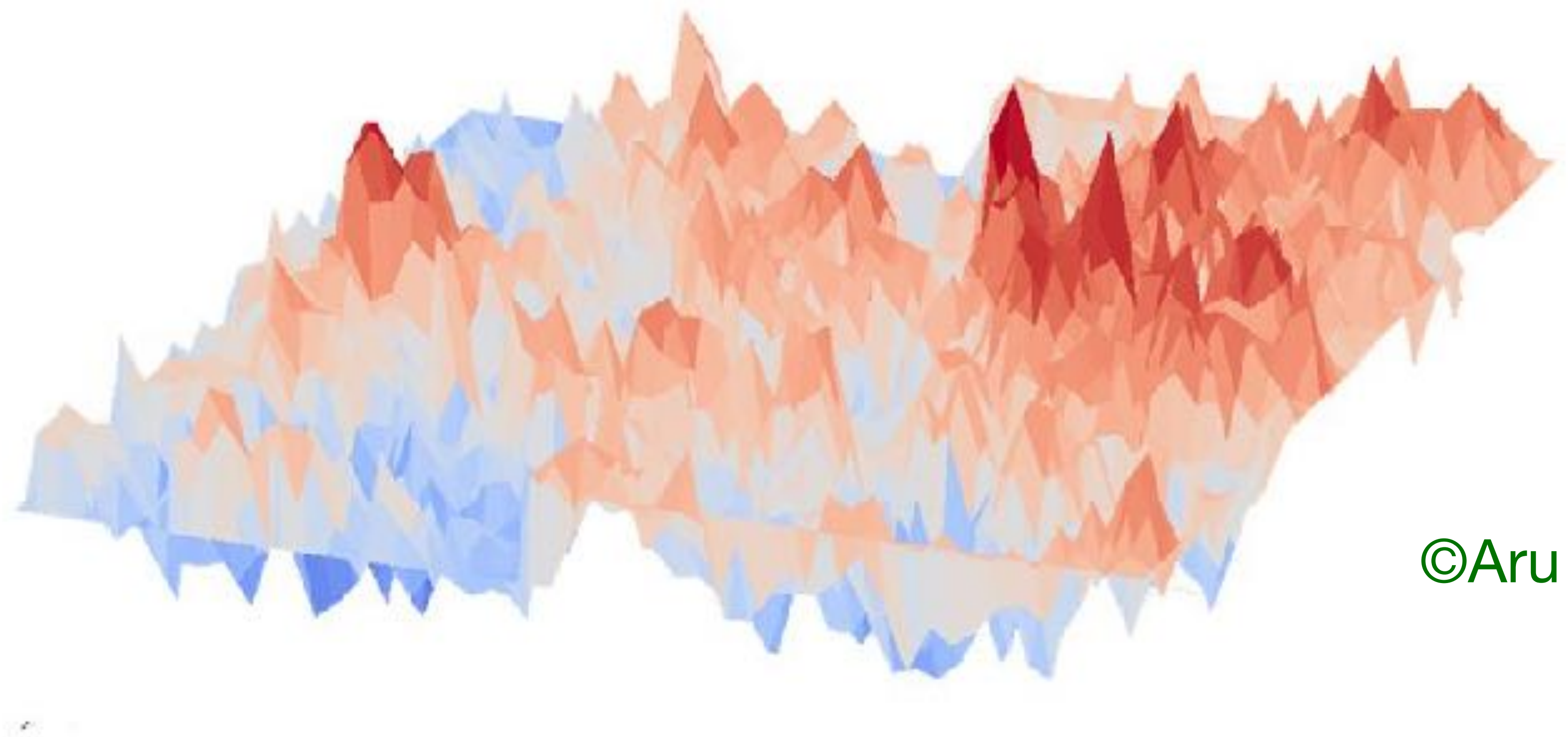

Thanks!