Pressure regularity for weak solutions of the Euler equations in the presence of boundaries

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Description of an inviscid fluid

One of the main differential equations of fluid mechanics are the incompressible Euler equations

$$\partial_t u + (u \cdot \nabla) u + \nabla p = 0, \quad \nabla \cdot u = 0.$$

If we project onto the divergence-free vector fields, we are left with the equation

$$\partial_t u + \mathbb{P}_h [u \cdot \nabla u] = 0.$$

The pressure can then be recovered as follows

$$-\Delta p = (\nabla \otimes \nabla) : (u \otimes u),$$

which can be obtained by taking the divergence of the Euler equations.

Description of the problem

We assume the no-normal flow boundary condition

$$(u \cdot n)|_{\partial\Omega} = 0.$$

Problem: Given that the velocity field has regularity $C^{0,\alpha}$, what regularity does the pressure have?

The reason we measure the regularity in Hölder spaces is that they play a fundamental role in turbulence theory.

One can formally derive the boundary condition from the Euler equations in order to get

$$\partial_n \left(p + (u \cdot n)^2 \right) = u \otimes u : \nabla n.$$

If $u \in C^{0,1/2+}$ then $\partial_n (u \cdot n)^2 = 0$, but we are also interested in the low regularity setting.

Previous work

- Onsager's conjecture in the presence of boundaries: [Robinson, Rodrigo, and Skipper, 2018; Bardos and Titi, 2018; Bardos, Titi, and Wiedemann, 2019]
- Pressure regularity results: [Silvestre, 2010; Constantin, 2014; Colombo, De Rosa, and Forcella, 2020; Bardos and Titi, 2021; De Rosa, Latocca, and Stefani, 2022]

The role of the pressure

Theorem (Bardos-DWB-Titi)

Let Ω *be a bounded domain with* $\partial\Omega \in C^2$*, and* $u \in L^3_t(C^{0,\alpha}_x)$ *be a weak solution of the Euler equations such that* $(u \cdot n)|_{\partial\Omega} = 0$ *(no-normal flow), then there exists a unique pressure* $p \in L^{3/2}_t(C^{0,\alpha}_x)$ *solving the equation*

$$-\Delta p = (\nabla \otimes \nabla) : (u \otimes u). \tag{1}$$

Moreover, the pressure satisfies the following Neumann boundary condition

$$\partial_n \Big(p + (u \cdot n)^2 \Big) \Big|_{\partial\Omega} = u \otimes u : \nabla n. \tag{2}$$

A new boundary condition

We show that it is necessary to include the term $\partial_n (u \cdot n)^2$, i.e. one has to use the very weak formulation of the boundary condition.

Example (Bardos-DWB-Titi)

Consider the following Weierstrass-type flow on the half-space

$$u_1(x, y, t) = -\sum_{k=0}^{\infty} 2^{-\alpha k} \sin(2^k \pi x) \cos(2^k \pi y), \tag{3}$$

$$u_2(x, y, t) = \sum_{k=0}^{\infty} 2^{-\alpha k} \cos(2^k \pi x) \sin(2^k \pi y). \tag{4}$$

We have that:

- $u \in C^{0,\alpha}(\Omega)$, $(u \cdot n)|_{\partial\Omega} = 0$ and u is divergence-free.
- $\partial_n (u \cdot n)^2|_{\partial\Omega} \notin \mathcal{D}'(\partial\Omega)$.

Conclusion

- We prove a pressure regularity result that is necessary for proving energy conservation of weak solutions
- It is necessary for the regularity proof to introduce a new weak formulation of the boundary condition
- We construct an example of a flow where the use of this formulation of the boundary condition is essential

Bibliography I

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