The Mathematics of Deep Learning
Lecture 7: Limitations of Deep Learning

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LMS Invited Lecture Series

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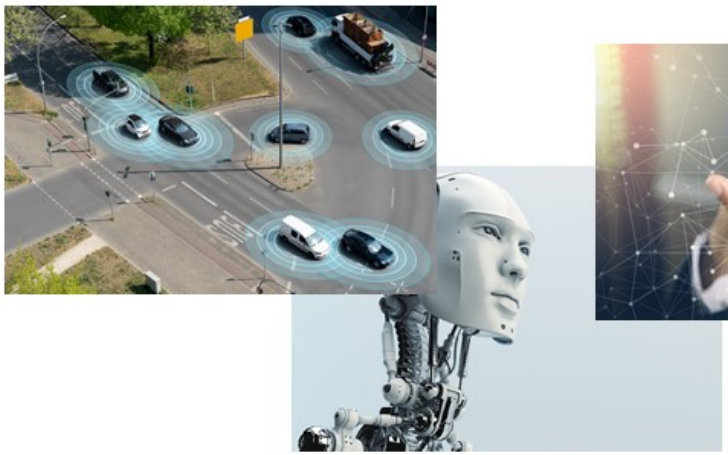
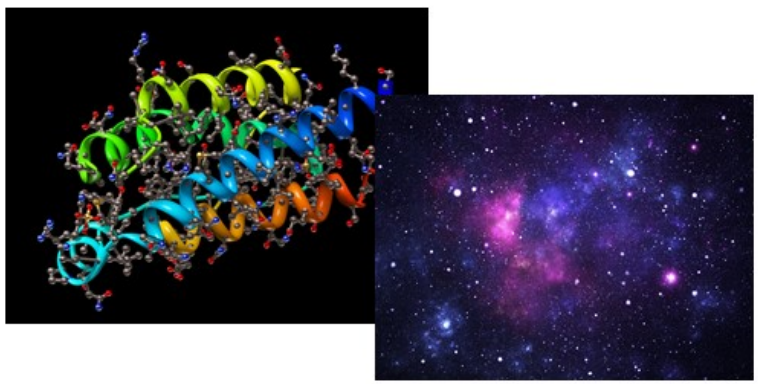
Neural Network Successes





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- tremendous *success* for problems in scientific computing,
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Problems/Limitations:

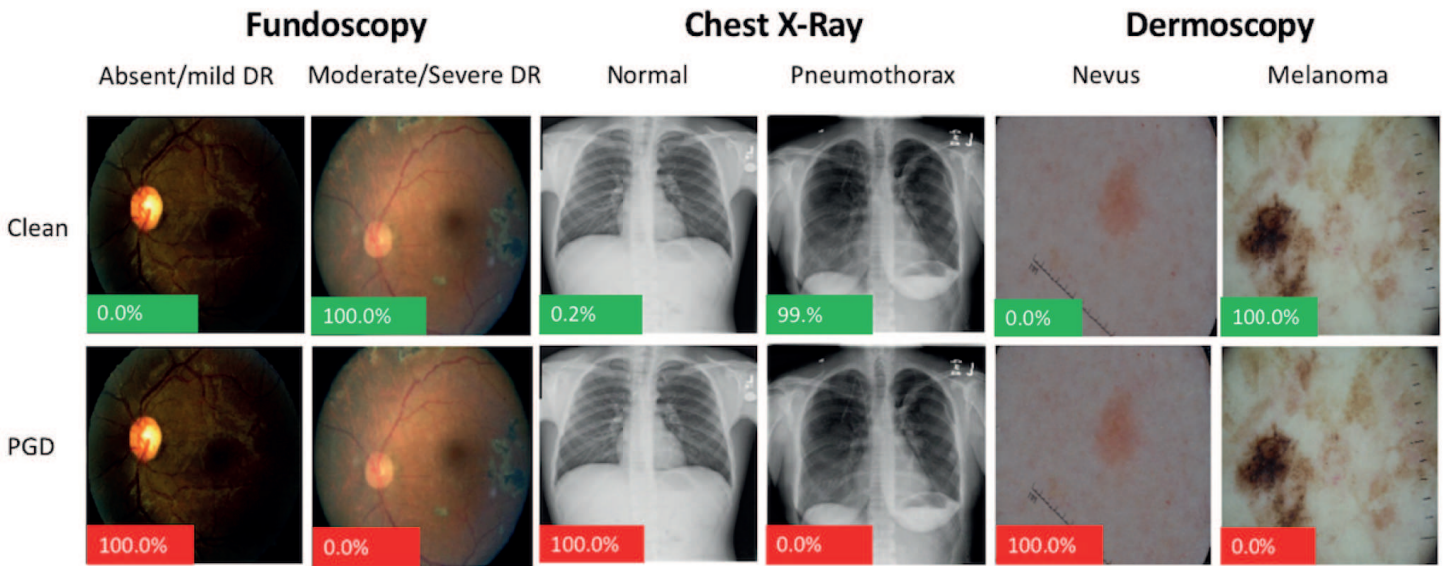
- Robustness
- Explainability
- Severe dependence on data
- Specific task
- Reasoning
- ....

Robustness

Requirements:

- Robustness problems should be immediately detectable or avoidable.
- Heuristic approaches do not satisfy certification standards.

Is complete robustness at all possible?



Source: Finlayson, Chung, Kohane, Beam, Adversarial Attacks Against Medical Deep Learning Systems, arXiv:1804.05296

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Explainability

Requirements:

- It should be possible to ask any question about a decision.
- The answer should reason as a human.

Is this achievable by connecting deep learning to natural language processing?

Severe Dependence on Data

Amount of Data:

- Many applications do not have large amounts of training data.
- Methods such as data augmentation do not compensate this fully.

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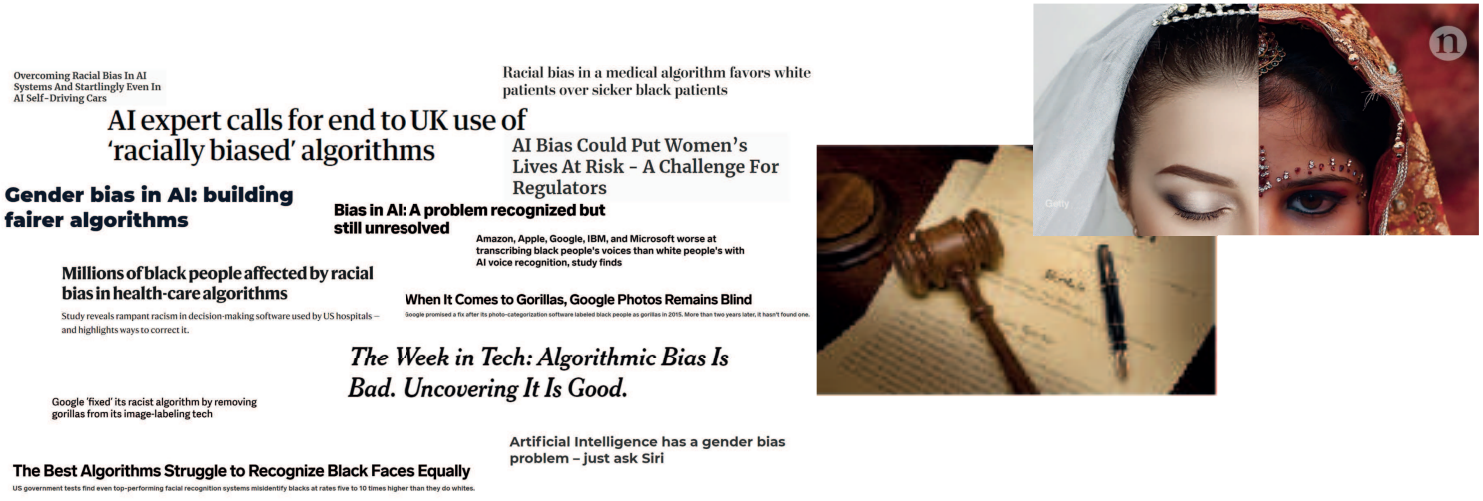
Problematic Data:

- Training data can unknowingly be biased.
- Uncertainty in the data can occur.

Can these problems be tackled at all?



Specific Task and Reasoning

Requirements:

- It should be possible to train for multiple tasks.
- The neural network should also be able to reason.
- Ideally, lifelong learning should be possible.

Are there fundamental limitations that constrain us?

An Even More Serious Problem

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- convergence of training algorithms
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- required sample complexity
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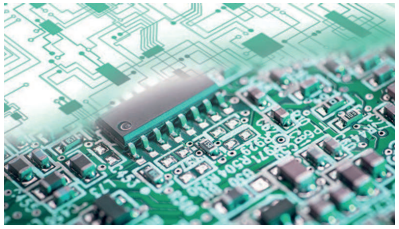

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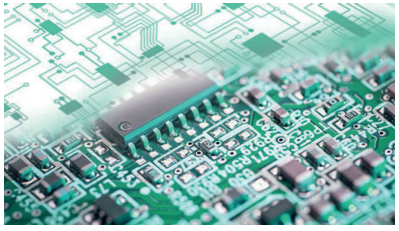

Theory-to-Practice Gap!

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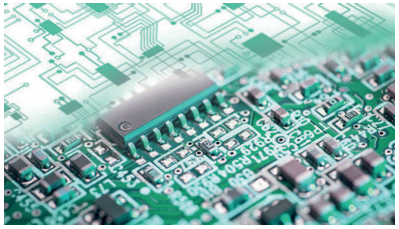

Theory-to-Practice Gap!

Goal: Examine the boundaries imposed by digital computations!

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What can actually be computed?

Computability on Digital Machines (informal):

A *computable problem (function)* is one for which the input-output relation can be computed on a digital machine for any given accuracy.

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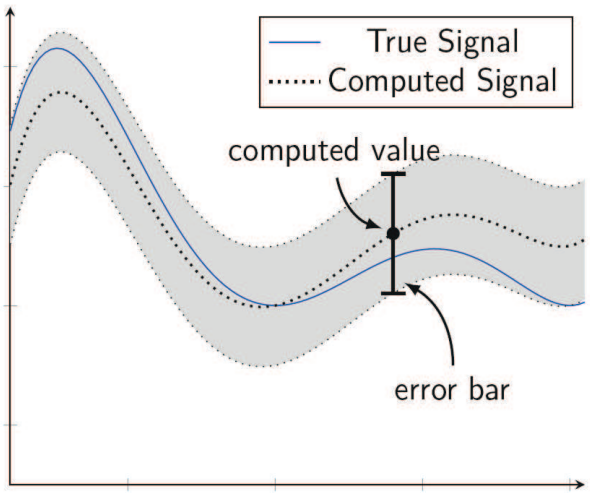
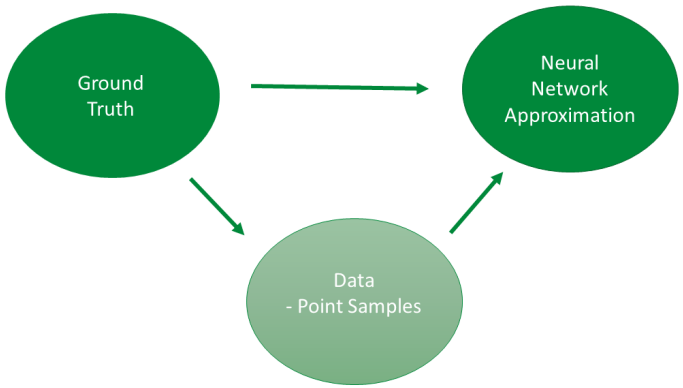
Questions:

- Is the underlying problem feasible?
  $\rightarrow$ *Computability of the ground truth*
- Are the neural networks computable?
  $\rightarrow$ *Computability of the network*
- Can the neural networks be found with the minimization problem?
  $\rightarrow$ *Computability of the mapping from data to approximation*



Why does Computability Matter?

- What is the *best* we can hope for?
  - Non-computability of the *ground truth*
    $\rightarrow$ *No approximation scheme*
  - Non-computability of the *network*
    $\rightarrow$ *Despite existence, network may not be computable*
  - Non-computability of the *mapping* from data to approximation
    $\rightarrow$ *Learning not feasible*



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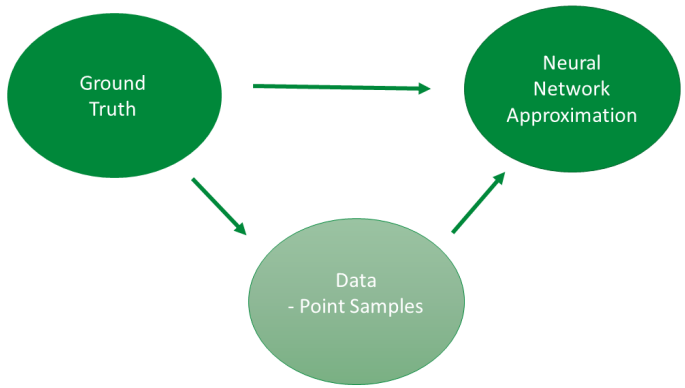

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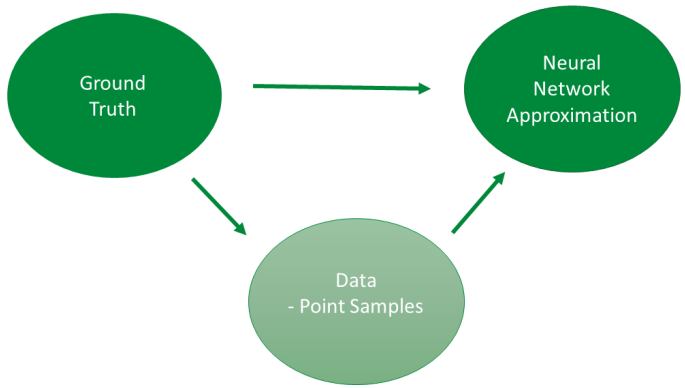

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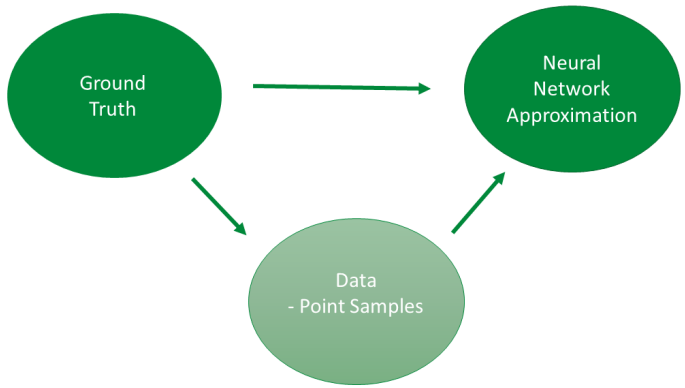

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Non-computable problems can be tackled successfully in practice, if limited precision suffices!

But we have no guarantees of correctness!

Theory of Computation

Turing machine $\longleftrightarrow$ Abstract idealization of *digital computer*

$\downarrow$ $\downarrow$

Discrete problems Scientific computing: *Continuous problems*

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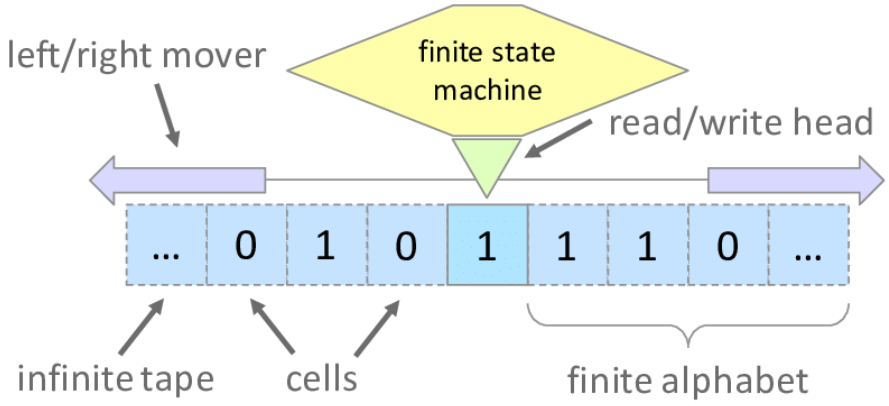
Discrete problems Scientific computing: *Continuous problems*

Definition:

"A Turing machine is a mathematical model of computation that defines an abstract machine that manipulates symbols on a strip of tape according to a table of rules."



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Computable Numbers

Definition:

A *computable real number* r is one for which there is a Turing machine with the following property: Given $n \in \mathbb{N}$ on its initial tape, it terminates with a rational number q such that $|r - q| \leq 2^{-n}$.

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Properties:

- The set of computable real numbers $\mathbb{R}_c$ is countable and dense in $\mathbb{R}$.
- Irrational numbers can be computable, like e and π.
- There is no algorithm that decides whether two computable real numbers are equal.

Computable Functions

Definition:

A function $f : \mathbb{R} \to \mathbb{R}$ is *computable*, if there exists an algorithm (Turing machine) Γ_f, which gives for all computable $x \in \mathbb{R}_c$ and all $n \in \mathbb{N}$ an approximation to $f(x)$ with

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Remarks:

- Minimal requirement for an algorithmic computation of a problem:
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Example of Non-Computable Functions:

- Constant functions equal to non-computable numbers
- Discontinuous functions

Specific Example

Consider the modeling of a physical system S on a digital computer.

- Assume a mathematical model S_{mod} for S describes the physical process and,allows to predict the output of S for any given input.

How well does S_{mod} describe the real physical process S?

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Control $\|\tilde{y}_{pred} - y_{pred}\|$ and y_{pred} of S_{mod} algorithmically on the computer!

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Control $\|\tilde{y}_{pred} - y_{pred}\|$ and y_{pred} of S_{mod} algorithmically on the computer!

Otherwise...

- ...the calculated solution $\tilde{y}_{\text{pred}}$ might be far from y_{pred} and comparing measurements of S with $\tilde{y}_{\text{pred}}$ becomes meaningless!
- ...no information about the quality of the mathematical model!

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A Large Problem Class

Computable Ground Truth?

Question: Is the problem (function) we try to approximate computable?

$\rightarrow$ *No good approximation if the function is non-computable.*

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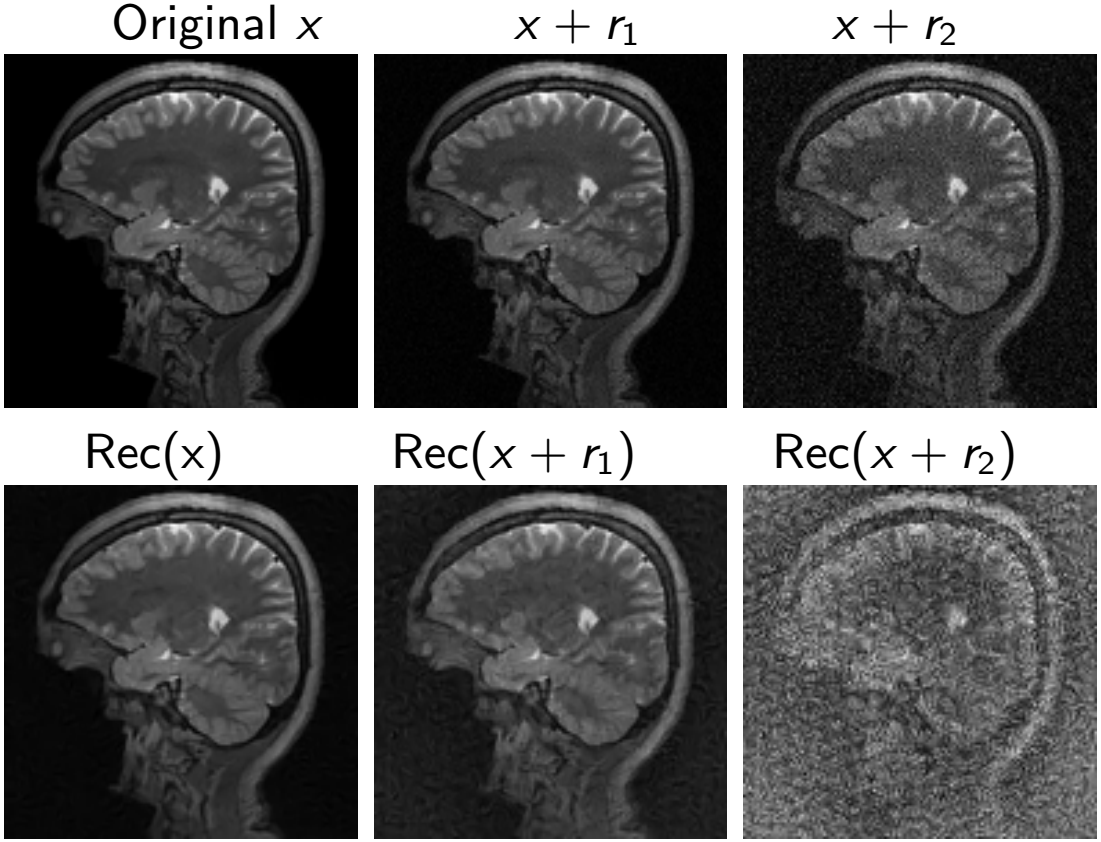
Example (MRI): Let $A \in \mathbb{C}^{m \times N}$ and $y = Ax + e \in \mathbb{C}^m$, recover $x \in \mathbb{C}^N$.



Source: Gottschling, Antun, Adcock, Hansen. The troublesome kernel: why deep learning for inverse problems is typically unstable. (arXiv:2001.01258)

Inverse Problem in Imaging

Recall:

Given $A \in \mathbb{C}^{m \times N}$ and $y = Ax + e \in \mathbb{C}^m$ of $x \in \mathbb{C}^N$, recover x.

Properties:

- $A \in \mathbb{C}^{m \times N}$ sampling operator, typically $m < N$ or even $m \ll N$
- successful approaches:
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Optimization Problem:
Given $A \in \mathbb{C}^{m \times N}$ and measurements $y \in \mathbb{C}^m$, solve

$$\arg \min_{x \in \mathbb{C}^N} \|x\|_{\ell^1} \text{ such that } \|Ax - y\|_{\ell^2} \leq \varepsilon, \quad \varepsilon > 0.$$

Solution Set

Solution Set:

For $A \in \mathbb{C}^{m \times N}$ and $y \in \mathbb{C}^m$ let

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Fundamental Questions:

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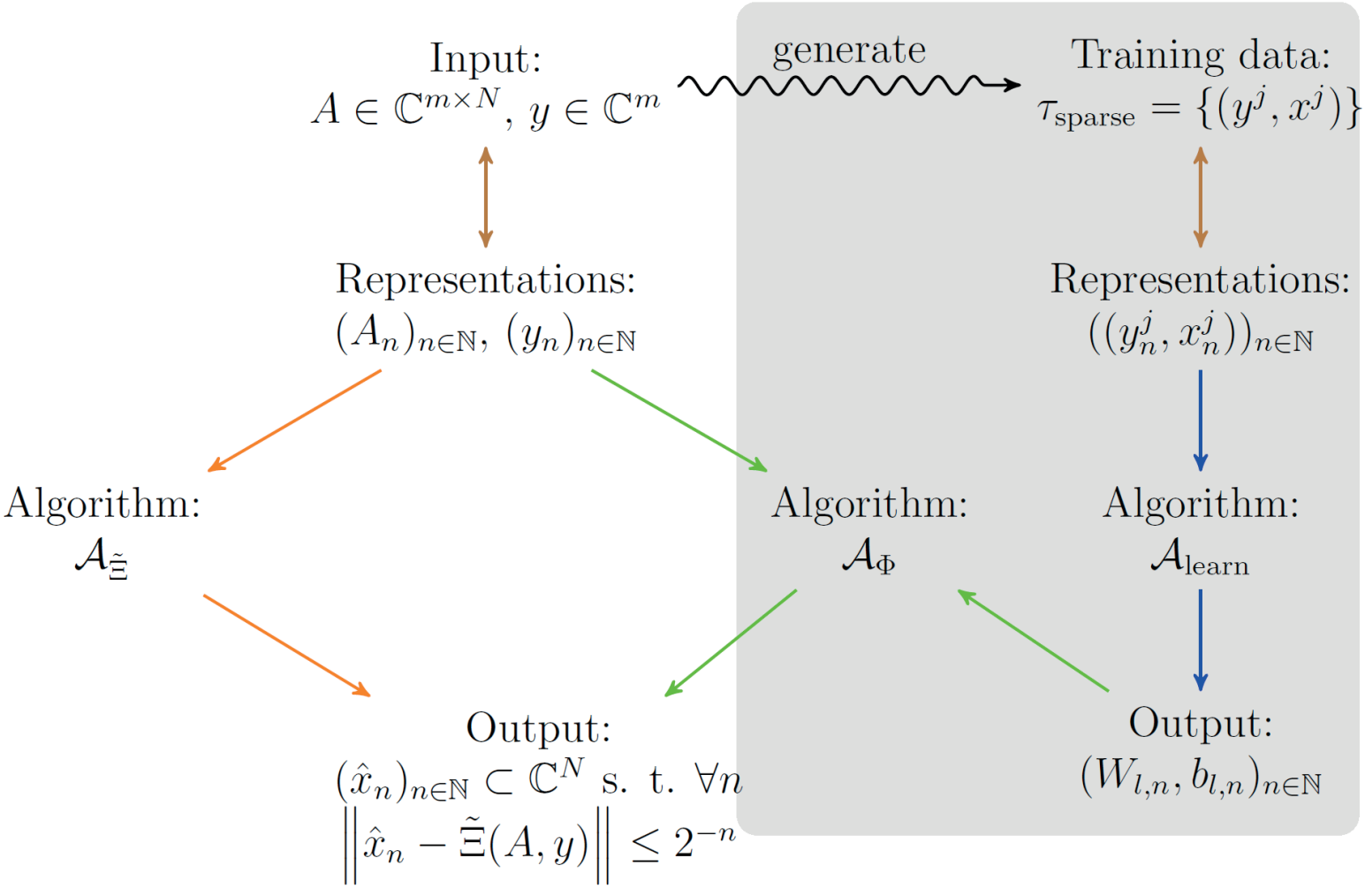
What are inherent restrictions of deep learning (performed on digital hardware)?

Are we *missing the correct tools and algorithms* to train neural networks adequately on digital machines or do *such algorithms not exist at all*?

Connecting the Dots



deep learning approach

A Bit Disappointing News

Non-Computability of Finite Dimensional Inverse Problems

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Theorem (Boche, Fono, K; 2022):

The function $\Psi : \mathbb{C}^{m\times N} \times \mathbb{C}^m \to \mathbb{C}^N$ for fixed parameters $\epsilon \in (0, 1)$, $N \geq 2$, and $m < N$, is *not computable*.

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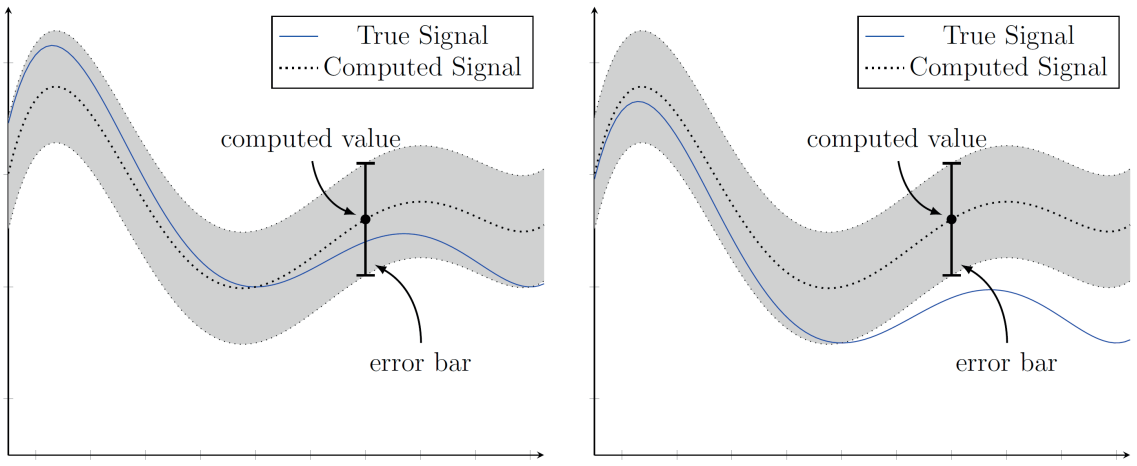
Illustration of the Problem:



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Banach-Mazur Computability

Definition:

A function $f : I \to \mathbb{R}^n_c$, $I \subset \mathbb{R}^m_c$, is said to be *Banach-Mazur computable*, if f maps computable sequences $(t_n)_{n\in\mathbb{N}} \subset I$ onto computable sequences $(f(t_n))_{n\in\mathbb{N}} \subset \mathbb{R}^n_c$.

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Lemma:

If a function is not Banach–Mazur computable, then it is not computable with respect to any other reasonable notion of computability including (Borel-)Turing computability.

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Remark:
The theorem holds even for Banach-Mazur computability.

Towards the Proof of...

Theorem (Boche, Fono, K; 2022):
The function $\Psi : \mathbb{C}^{m\times N} \times \mathbb{C}^m \to \mathbb{C}^N$ for fixed parameters $\epsilon \in (0,1)$, $N \geq 2$, and $m < N$, is *not (Banach-Mazur) computable*.

Recall the Solution Set:
For $A \in \mathbb{C}^{m\times N}$ and $y \in \mathbb{C}^m$ let

$$\Psi(A,y) := \arg\min_{x\in\mathbb{C}^N} \|x\|_{\ell^1} \text{ such that } \|Ax-y\|_{\ell^2} \leq \varepsilon.$$

Non-Computability Lemma

Lemma:

Suppose there are computable sequences $(\omega_n^1)_{n\in\mathbb{N}}, (\omega_n^2)_{n\in\mathbb{N}} \subset \mathbb{C}^{m\times N} \times \mathbb{C}^m$ satisfying the following conditions:

(a) There are sets $S^1, S^2 \subset \mathbb{C}^N$ and $\kappa > 0$ such that $\Psi(\omega_n^j) \subset S^j$ and

$$\inf_{x_1\in S^1, x_2\in S^2} \|x_1 - x_2\|_{\ell^2} > \kappa$$

.

(b) There is an $\omega^* \in \mathbb{C}^{m\times N} \times \mathbb{C}^m$ such that $\Psi(\omega^*)$ is computable and

$$\left\|\omega_n^j - \omega^*\right\|_{\ell^2} \leq 2^{-n} \quad \text{for all } n \in \mathbb{N}, j = 1, 2.$$

Then the map $\Psi : \mathbb{C}^{m\times N} \times \mathbb{C}^m \to \mathbb{C}^N$ is *not computable*.

Previous Work

Oracle Turing Machines

Closest Work:

- Classification: (Bastounis, Hansen, Vlacic; 2021)
- Inverse Problems: (Colbrook, Antun, Hansen; 2021)

Those approaches employ a so-called Oracle Turing machine.

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"The *Oracle Turing machine* is an extension of the regular Turing machine with an additional oracle tape that, given a real or complex number, grants the Turing machine access to a rational approximation of arbitrary — or more exactly demanded — precision."

Oracle Turing Machines

Closest Work:

- Classification: (Bastounis, Hansen, Vlacic; 2021)
- Inverse Problems: (Colbrook, Antun, Hansen; 2021)

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Relation to Turing Machines:

- Oracle Turing machines exceed the potential of regular Turing machines.
- Oracle Turing machines omit the problem that not all complex numbers can be approximated.

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A More Detailed Comparison

Question Considered in Previous Work:

- Given *arbitrarily accurate approximations to any complex number* such as, in particular, the training samples.

Under these conditions:

- Do there exist algorithms that yield accurate neural networks?
- Are there computational barriers in the training of neural networks?

$\rightsquigarrow$ *Is this realistic for autonomous digital computations?*

A More Detailed Comparison

Example:

Consider the computation of a function f through an Oracle Turing machine.

- Assume that
  - on the *non-computable numbers*, f exhibits a behaviour that prevents the computation by means of a Turing machine,
  - restricted to the *computable numbers*, the function f is well-behaved and can be computed by a Turing machine.

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(1) f is not computable by an Oracle Turing machine.

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$\rightsquigarrow$ *Thus non-computability of a function on a Turing machine is a strictly stronger result than non-computability on an Oracle Turing machine!*

$\rightsquigarrow$ *This raises some doubts whether Oracle Turing machines are an adequate model for autonomous real-world computations!*

Consequences

Some Thoughts on the Result

Corollary:

- Ψ can *not be approximated* to arbitrary accuracy on digital computers.
- *No algorithm exists*, which on digital hardware derives neural networks Φ_A approximating $\Psi(A, \cdot)$ for any given accuracy and all $A \in \mathbb{C}^{m \times N}$.
- The output of trained neural networks *not reliable (no guarantees)*.

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General Barrier:

This barrier on the capabilities of neural networks for finite-dimensional inverse problems is caused by *a combination* of the following two separate aspects:

- The mathematical structure and properties of *finite-dimensional inverse problems*.
- The mathematical structure and properties of *Turing machines* and thereby also of *digital machines*.

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- The problem is *inherently connected* to digital computations.
- Deep learning has still impressive results, but a *necessary caution* is on order.
- This result could point towards why *instabilities* and *non-robustness* occurs for deep neural networks.
- A *subtlety*: This does not necessarily mean that an algorithm yielding a network with a required, fixed precision can not be obtained. However, any prescribed precision may require different Turing machines, i.e., different algorithms.

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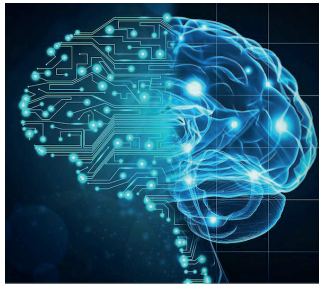
Other Models of Computations:

- New emerging hardware
  - *Neuromorphic computing*: Elements of computer modeled after systems in the human brain and nervous system.
  - *Biocomputing*: Living cells as the substrate for performing human-defined computations
- Different models of computation required



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Key Future Question:

Does the non-computability result also hold for different computation models such as analog computers as well?

Some Final Thoughts...

Conclusions

Some General Limitations:

- Robustness (Adversarial attacks, etc.)
- Explainability
- Severe dependence on data (bias, requires huge amount, ...)
- Specific task
- Reasoning

Computability

- A *Turing machine* is a suitable model for digital computers.
- *Computability* needs to be studied.
- We focus on *finite-dimensional inverse problems*.

There does not exist an algorithm which performs the training of a neural network on digital hardware for any given accuracy!

THANK YOU!

References available at:

`www.ai.math.lmu.de/kutyniok`

Survey Paper (arXiv:2105.04026):

Berner, Grohs, K, Petersen, *The Modern Mathematics of Deep Learning*.

Check related information on Twitter at:

`@GittaKutyniok`

Upcoming Book:

- P. Grohs and G. Kutyniok, eds.
  Mathematical Aspects of Deep Learning
  Cambridge University Press, to appear.

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