Neural Controlled Differential Equations

Mathematical Institute

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Oxford Mathematics

Overview

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Applications to traditional mathematical modelling (nance, physics, engineering, ...), and to modern machine learning (time series, generative modelling).

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Programme

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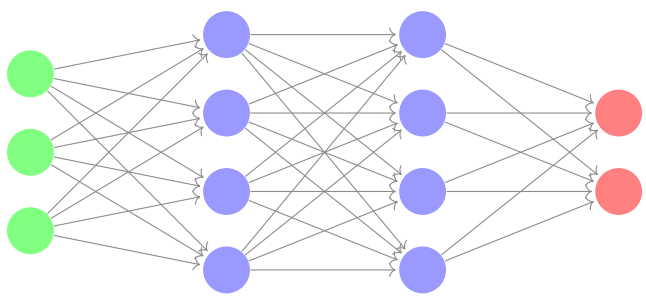

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Classical example of a `neural' di erential equation: the SIR model.

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} s(t) \\ i(t) \\ r(t) \end{pmatrix} = \begin{pmatrix} -\beta s(t) i(t) \\ \beta s(t) i(t) - \kappa i(t) \\ \kappa i(t) \end{pmatrix}$$

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which rearranges to

$$y_{t_{n+1}} = y_{t_n} + \quad t\, f(t; y_{t_n}):$$

This is now the same as a residual network, one of the main neural network architectures.

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From traditional mathematical modelling: neural networks are very high capacity models, easy to train; the NDE formalism means they easily augment traditional techniques.

From machine learning: di erential equations are good priors on model space, able to handle irregular data, memory e cient, give theoretical grounding to many neural networks.

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- Physical (nancial, . . .) modelling, for which a di erential equation is explicitly desired.
- Generative modelling: continuous normalising ows; neural SDEs.
- Time series applications: data may arrive irregularly sampled, partially observed and so on.

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Want a way to incorporate incoming information.

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Changes in X provoke changes in y.

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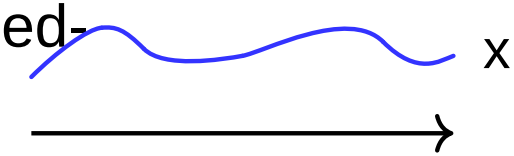

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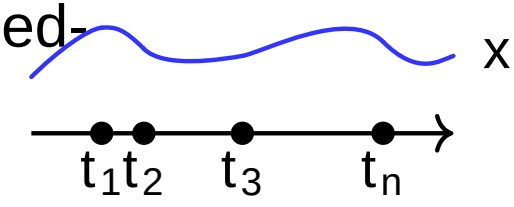

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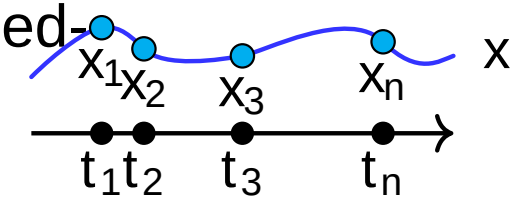

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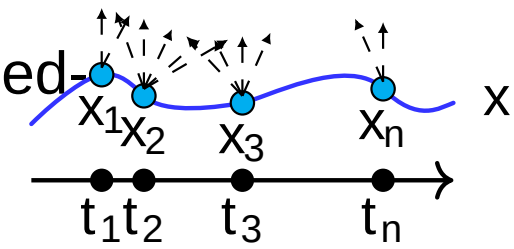

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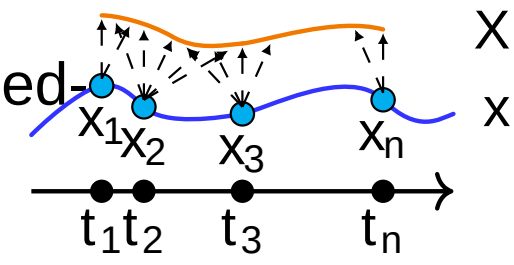

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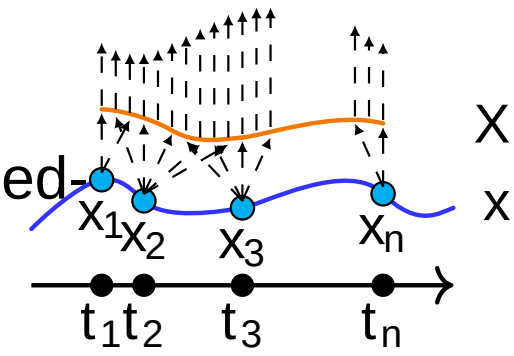

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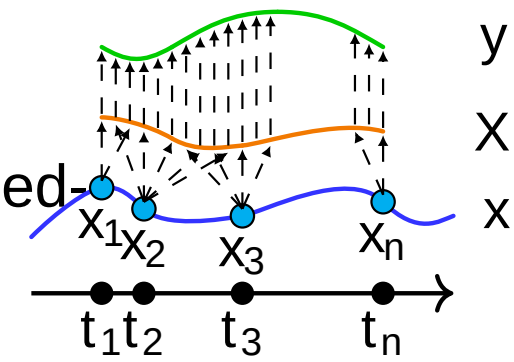


Neural controlled di erential equations

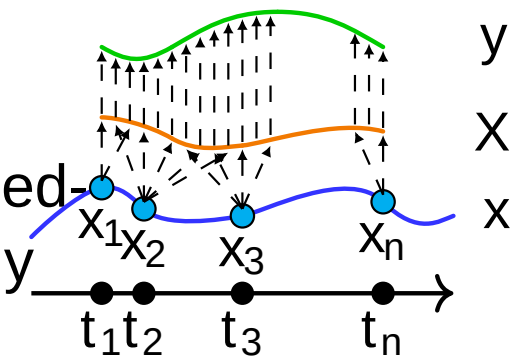
Irregular time series

Observe $x = ((t_0; x_0); : : : ; (t_n; x_n))$ with $t_i \in R$ and $x_i \in R^v$.
Let $X : [t_0; t_n] \to R^{v+1}$ interpolate this data, so $X(t_i) = (t_i; x_i)$.

Learn functions $, f$ and a linear map $`$ such that

$$y(t_0) = (t_0; x_0); \quad y(t) = y(t_0) + \int_{t_0}^{t} f(y(s))\, dX(s);$$

and the output is either $`(y(t_n))$ or $`(y(t))$.



and f are arbitrary neural networks (feed-forward, ...), and y is hidden state. Directly analogous to an RNN "$y_{n+1} = f(y_n; x_n)$".

We call this aneural controlled di erential equation.

Neural controlled di erential equations

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- Spoken audio.
- Weather data, e.g. temperature and pressure as they change over time.
- Physics: the movement of a double pendulum.

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Neural CDEs demonstrate state-of-the-art performance.

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Irregular time series

Model	Test Accuracy (mean std, computed across ve runs)						Memory usage
	30% dropped		50% dropped		70% dropped		
GRU-ODE	92.6%	1.6%	86.7%	3.9%	89.9%	3.7%	1.5
GRU- t	93.6%	2.0%	91.3%	2.1%	90.4%	0.8%	15.8
GRU-D	94.2%	2.1%	90.2%	4.8%	91.9%	1.7%	17.0
ODE-RNN	95.4%	0.6%	96.0%	0.3%	95.3%	0.6%	14.8
Neural CDE (ours)	98.7%	0.8%	98.8%	0.2%	98.6%	0.4%	1.3

Comparison to control theory problems (speaking very broadly):

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Comparison to control theory problems (speaking very broadly):

- Control theory: System f is fixed; try to find optimal X producing a desired response y.
- (Neural) CDEs: Input X is fixed; try to find optimal f producing a desired response y.

Neural stochastic di erential equations

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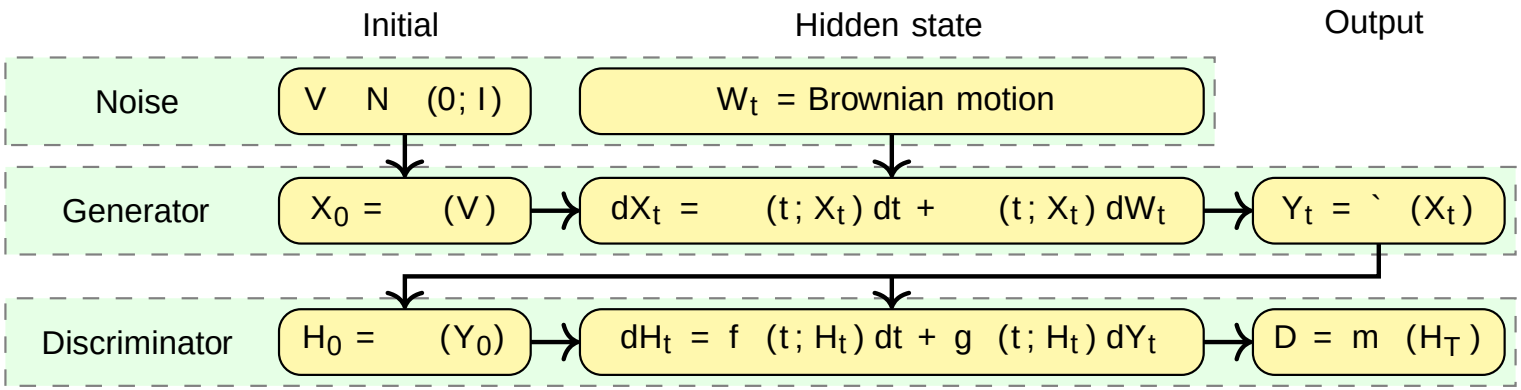

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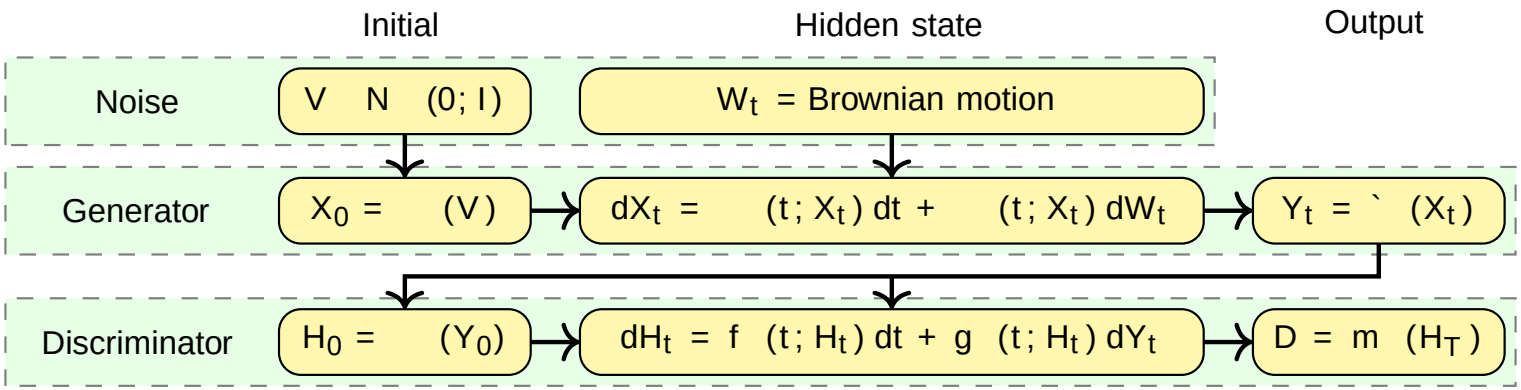
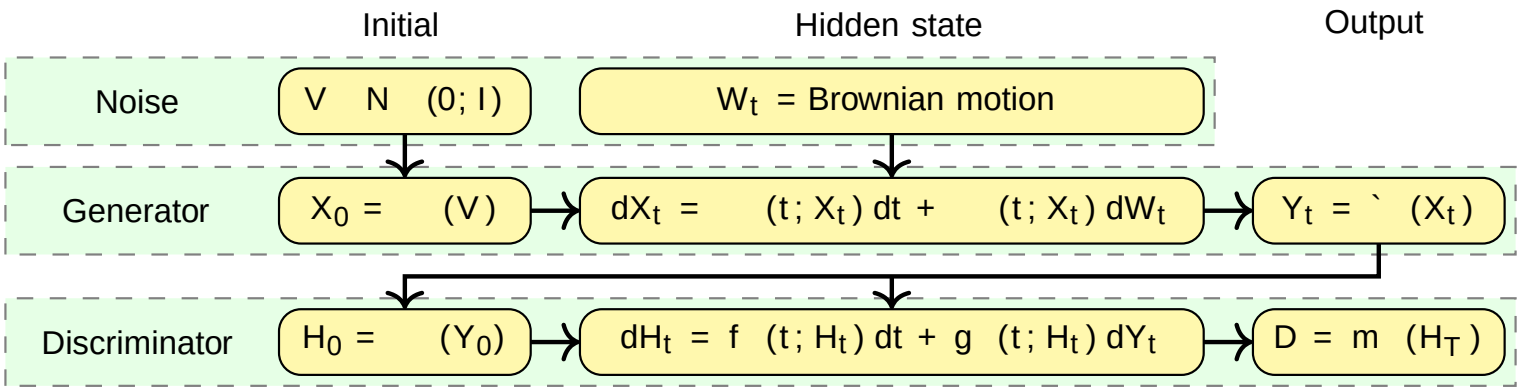


Neural SDE / CDE form a generator/discriminator pair.

Neural stochastic di erential equations



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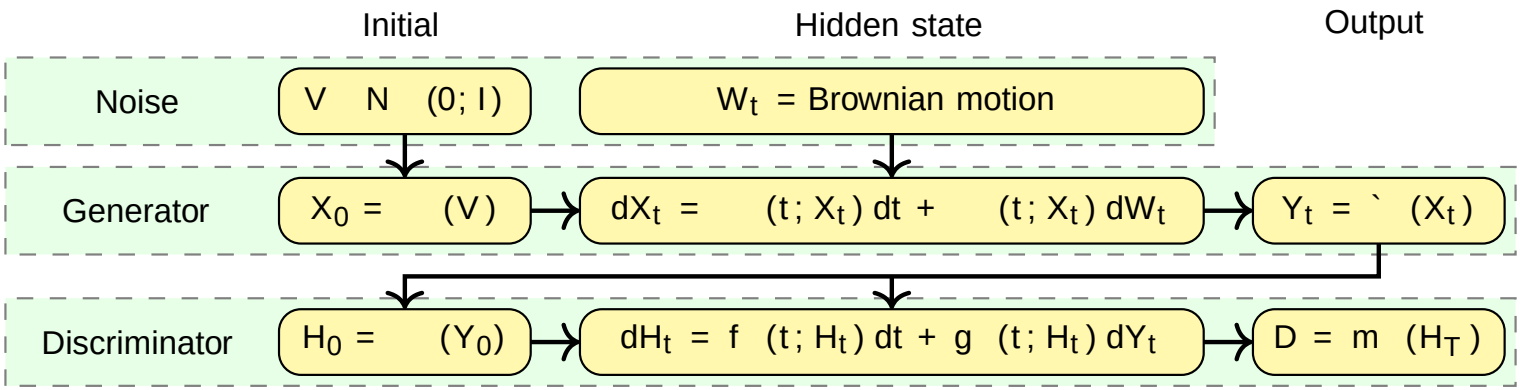

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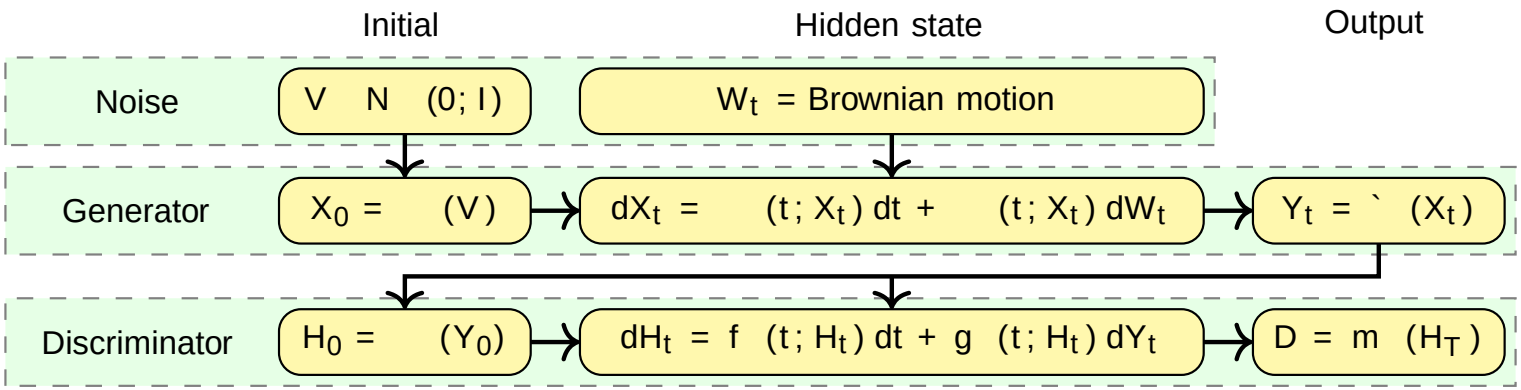

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We've come a long way from models like the Heston model...

$$dS_t = \quad S_t\,dt + \sqrt{\ }_t S_t\,dW_t^1$$
$$d\ _t = \quad (\qquad _t)\,dt + \sqrt{\ }_t\,dW_t^2$$

Neural stochastic di erential equations

Google/Alphabet Stocks:

Metric	Neural SDE		CTFP		Latent ODE	
Classi cation	0.357	0.045	0.165	0.087	0.000239	0.000086
Prediction	0.144	0.045	0.725	0.233	46.2	12.3
MMD	1.92	0.09	2.70	0.47	60.4	35.8

Beijing Air Quality:

Metric	Neural SDE		CTFP		Latent ODE	
Classi cation	0.589	0.051	0.764	0.064	0.392	0.011
Prediction	0.395	0.056	0.810	0.083	0.456	0.095
MMD	0.000160	0.000029	0.00198	0.00001	0.000242	0.000002

SGD dynamics:

Metric	Neural SDE		CTFP		Latent ODE	
Classi cation	0.507	0.019	0.676	0.014	0.0112	0.0025
Prediction	0.00843	0.00759	0.0808	0.0514	0.127	0.152
MMD	5.28	1.27	12.0	0.5	23.2	11.8

Software

We've got standardised software for solving neural di erential equations.

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PyTorch:

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Autodi erentiable, GPU-capable, composable.

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Widespread applications to improve traditional mathematical modelling (nance, physics, engineering, :), and to modern machine learning problems (time series, generative modelling).

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Neural di erential equations are a powerful extension to this paradigm.

Widespread applications to improve traditional mathematical modelling (nance, physics, engineering, :), and to modern machine learning problems (time series, generative modelling).

Convince the rest of the world that they should be using neural di erential equations too.

References

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torchsde : https://github.com/google-research/torchsde
torchcde : https://github.com/patrick-kidger/torchcde

Di erentialEquations.jl:
https://github.com/SciML/DifferentialEquations.jl

Final notes

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Any questions now?