Simulating spot and equity option markets using rough path signatures

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Deep Hedging

Problem setting

- Deep Hedging (DH) aims to automate hedging a portfolio of derivatives under market frictions.
- Calibration of the DH agent requires *large amounts* of realistic daily market data for an underlying which is not available.

\[
G: \mathbb{R}^p \times (\mathbb{R}^n)^{m+1} : \to \mathbb{R}^{nm+1} \\
R_{t+1}, \sigma_{t+1} = G(Z_{t+1}, R_t, \ldots, R_{t-p+1}, \sigma_t, \ldots, \sigma_{t-p+1})
\]

where \(p \in \mathbb{N}, Z_{t+1} \sim \mathcal{N}(0, I_q)\), \(R_t\) is the 1-dimensional spot log-return at time \(t\), \(\sigma_t\) is a \(nm\)-dimensional grid (\(n\) strikes, \(m\) maturities) of discrete local volatilities (DLVs) at time \(t\), and \(G\) is unknown.
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We aim to mitigate the problem of limited market data by calibrating a realistic market simulator $G : \mathbb{R}^{p \cdot (nm+1) + q} \rightarrow \mathbb{R}^{nm+1}$

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How do we calibrate $G$?
Orthonormal path compression with signature cumulants

- Direct calibration of $G$ suffers from the curse of dimensionality; especially for dense grids $nm \gg 1$
- Instead, the grids of DLVs are compressed to an efficient lower-dimensional representation $\gamma_t := F(\sigma_t), F : \mathbb{R}^{nm} \to \mathbb{R}^l$ by using autoencoders
- Additionally, the components $(\gamma^1_t, \ldots, \gamma^l_t)$ are regularized to be statistically independent through space and time by leveraging a signature cumulants-based penalty (Bonnier et al., 2020)

![Illustration of a shallow DLV autoencoder](image-url)

**Figure:** Illustration of a shallow DLV autoencoder ($28 - 11 - 3 - 11 - 28$).
Figure: Compressed DLV times series \((\gamma_t)^T_{t=0}\)

Figure: Copula of \(\gamma_t\)

Figure: Copula of \(\Delta\gamma_t\)
Calibrating the market simulator with signatures

The market simulator is calibrated by minimizing the **conditional Sig-Wasserstein-1 loss** (Hao Ni et al., 2020):

\[
\mathcal{L}_\mu(\nu) := \mathbb{E}_{X_0, T} \left( \| \mathbb{E}_\mu(S(X_{t+T}|X_0, t) - \mathbb{E}_\nu(S(X_{t+T}|X_0, t) \|_2 \right)
\]

where \(X_0, T = (X_t)_{t \in [0, T]}\) is a linear interpolation of the discrete-time process \((R_t, \gamma_t)_{t \in \{0, \ldots, T\}}\), \(\mu\) is the empirical law, \(\nu\) the law induced by the market simulator, and the approximation of

- \(\mathbb{E}_\mu(S(X_{t+T}|X_0, t)\) boils down to a linear regression which can be estimated **robustly** via ridge regression / LASSO and cross-validation,
- \(\mathbb{E}_\nu(S(X_{t+T}|X_0, t)\) is a Monte Carlo approximation evaluated through generating samples \(X_{t, T} \sim \nu(\cdot|X_0, t)\).
Comparison of empirical distribution and invariant distribution induced by $G$

$$(R_{t+1}, \gamma_{t+1}) = G(Z_{t+1}, R_{t-p+1:t}, \gamma_{t-p+1:t})$$

**Figure:** Marginal distribution of $(R_t, \gamma_t)$

**Figure:** Autocorrelation function of $(R_t, \gamma_t)$

**Figure:** Cross-correls of $(R_t, \gamma_t)$
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