Method for Reconstruction of Electrical Properties inside Human Bodies using MRI

The University of Tokyo
Takaaki Nara

Joint work with M. Fushimi

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Background

▪ Electrical conductivity and permittivity imaging inside the human body
⇒ pathological information: breast carcinoma (tumor) exhibits conductivity changes up to a factor of 10. [Katscher, 2009]

⇒ Small tumor/cancer tissues can be detected

■ Diagnosis of malignant tissue [Joines *et al.*, 1994]

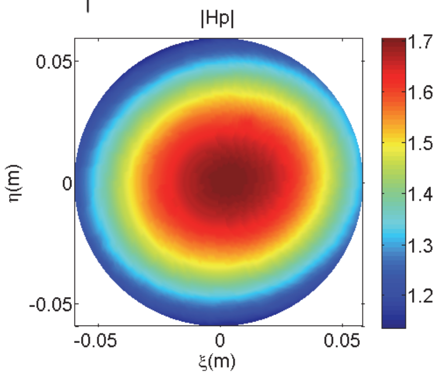
Tissue	σ_{normal}	$\sigma_{\text{malignant}}$	$\epsilon_{\text{r,normal}}$	$\epsilon_{\text{r,malignant}}$
Liver	0.49	0.66	62	66
Lung	0.49	0.82	77	69
Mammal	0.11	**0.78**	21	**69**

Magnetic resonance electrical property tomography (MREPT)

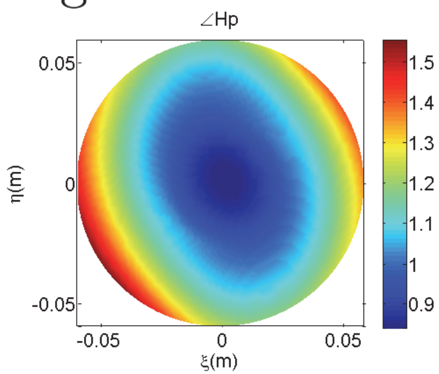
`Birdcage coil’ generates the rotating magnetic field with a frequency of e.g. 128MHz

<u>What is measured</u>: magnetic field inside the body

$|H^+|$

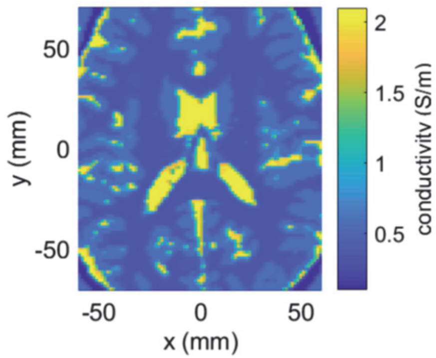


$\arg H^+$



<u>What should be reconstructed</u>:

Electric conductivity and permittivity



Governing equation: time-harmonic Maxwell's equations

Faraday's law	$\nabla \times \boldsymbol{E} = -i\omega\mu_0 \boldsymbol{H}$
Ampere's law	$\nabla \times \boldsymbol{H} = \gamma \boldsymbol{E} + \boldsymbol{J}$
Silver-Muller radiation condition	$\lim_{\lvert\boldsymbol{r}\rvert\to\infty} (\boldsymbol{H} \times \boldsymbol{r} - \lvert\boldsymbol{r}\rvert \boldsymbol{E}) = \boldsymbol{0}$

$\gamma = \sigma + i\omega\epsilon$: admittivity

conductivity (σ) permittivity (ϵ)

MREPT inverse problem: given $H^+ = \frac{1}{2}(H_x + iH_y)$ inside the body D ,

reconstruct $\gamma = \sigma + i\omega\epsilon$.

Conventional method

$$\left\{\begin{array}{l}\nabla \times \boldsymbol{E} = -i\omega\mu_0 \boldsymbol{H} \\ \nabla \times \boldsymbol{H} = \gamma \boldsymbol{E}\end{array}\right. \quad \text{in } D$$

Katscher,
IEEE Tr. Medical Imaging,
2009.

By eliminating $\boldsymbol{E}$, a non-linear PDE for γ is obtained:

$$(\nabla \times \boldsymbol{H}) \times \frac{\nabla \gamma}{\gamma} + i\omega\mu_0 \boldsymbol{H}\gamma = \Delta \boldsymbol{H}$$

By neglecting $\nabla\gamma$, a reconstruction formula is obtained:

$$\gamma = \frac{\Delta H^+}{i\omega\mu_0 H^+}$$

Objective

Derivation of an explicit reconstruction formula for inhomogeneous admittivity γ from the measured magnetic field H^+

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1. Derivation of a direct reconstruction formula for the 2D problem

2. Extension to the 3D problem

1. Derivation of a direct reconstruction formula

Assumptions:

1) $H_z = 0$ — Usual assumptions from the structure of `birdcage coils' aligned along the z-axis

2) $\partial_z H^+ = 0$ — γ does not change w.r.t the z-axis

- First, we consider the 2D problem and derive a direct reconstruction formula.
- Second, we remove Assumption 2) and propose an iterative algorithm.

Dbar equations derived from time-harmonic Maxwell's equations

Define the compelx derivatives: $\partial \equiv \frac{1}{2}(\partial_x - i\partial_y), \quad \bar{\partial} \equiv \frac{1}{2}(\partial_x + i\partial_y)$

Let $H^+ \equiv \frac{1}{2}(H_x + iH_y), \quad E^+ \equiv \frac{1}{2}(E_x + iE_y)$

Then, under the two assumptions, Maxwell's equations in D are written:

$$\begin{cases} \partial H^+ = \frac{i\gamma}{4} E_z & \text{Ampere's law} \\ \bar{\partial} E_z = \omega \mu_0 H^+ & \text{Faraday's law} \end{cases}$$

Ampere's law

$$4\partial H^{+} = i\gamma E_z \quad \Rightarrow \quad \gamma = \frac{4\partial H^{+}}{iE_z}$$

Complex derivative of the measured magnetic field

z-component of the electric field

Faraday's law Dbar equation

$$\bar{\partial} E_z = \omega\mu_0 H^{+}$$

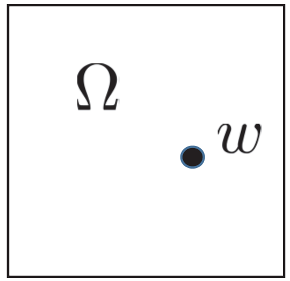
We use Faraday's equation to represent E_z in terms of H^{+}

Generalized Cauchy formula for Dbar equations

- $f \in C^1(\Omega) \cap C(\bar{\Omega})$, Ω : domain bounded by a Jordan contour
- f satisfies a Dbar equation: $\bar{\partial} f = g$ (inhomogeneous Cauchy-Riemann system)

➡ Generalized Cauchy formula

$$f(w) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(\zeta)}{\zeta - w} d\zeta - \frac{1}{\pi} \int\int_{\Omega} \frac{g(\zeta)}{\zeta - w} d\xi d\eta, \quad w \in \Omega$$



Faraday's law $\bar{\partial} E_z = \omega \mu_0 H^+$ Dbar equation

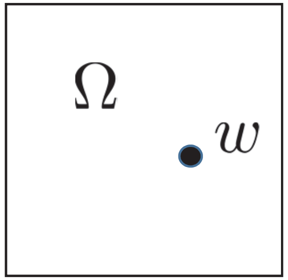
➡ The z-component of the electric field can be expressed as

$$E_z(w) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{E_z(\zeta)}{\zeta - w} d\zeta - \frac{\omega \mu_0}{\pi} \int\int_{\Omega} \frac{H^+(\zeta)}{\zeta - w} d\xi d\eta, \quad w \in \Omega$$

$$E_z(w) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{\boxed{E_z(\zeta)}}{\zeta - w} d\zeta - \frac{\omega\mu_0}{\pi}\int\int_{\Omega} \frac{H^+(\zeta)}{\zeta - w} d\xi d\eta, \quad w \in \Omega$$

Ampere's law $4\partial H^+ = i\gamma E_z$

$$\frac{4\partial H^+}{i\gamma}$$



If we give the boundary value of γ , the electric field E_z can be computed.

By substituting this into

$$\gamma(w) = \frac{4\partial H^+(w)}{iE_z(w)}$$

we arrive at the explicit reconstruction formula:

Direct reconstruction formula of the admittivity γ

Given H^+ in Ω and γ on $\partial\Omega$,

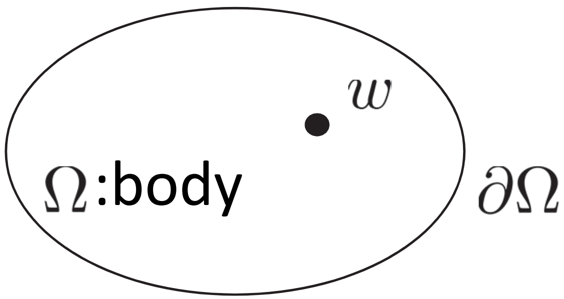
the adtmivity at an arbitrary point $w \in \Omega$ is reconstructed by

$$\gamma(w) = \frac{4\pi i \partial H^+(w)}{\int_{\partial\Omega} \frac{\frac{2}{\gamma(\zeta)}\partial H^+(\zeta)}{\zeta - w} d\zeta + \omega\mu_0 \int\int_{\Omega} \frac{H^+(\zeta)}{\zeta - w} d\xi d\eta}$$

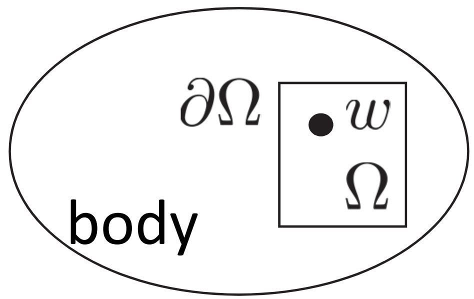
How to give the boundary value of γ

$$\gamma(w) = \frac{4\pi i \partial H^{+}(w)}{\int_{\partial\Omega} \frac{\frac{2}{\boxed{\gamma(\zeta)}}\partial H^{+}(\zeta)}{\zeta - w} d\zeta + \omega\mu_0 \int\int_{\Omega} \frac{H^{+}(\zeta)}{\zeta - w} d\xi d\eta}$$

Scenario 1: Take Ω as a whole body D and measure γ near the body surface

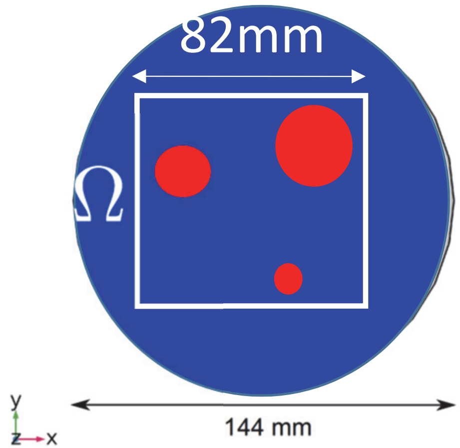


Scenario 2 : Take Ω as a local domain inside which a small cancer may exist and give a normal value of γ on $\partial\Omega$



Numerical simulations

- Three cylinders (conductivity = 1S/m, diameters = 10mm, 20mm, 30mm) in a cylinder (conductivity = 0.5 S/m).
- Forward solution was computed by an FEM software (COMSOL). f=123MHz (corresponding to 2.89T)
- Ω : 82mm x 82mm
- Discretization: pixel size = 1.4mm x 1.4mm



- Singular integral $$\int\int_{U_\epsilon(w)} \frac{H^+(\zeta)}{\zeta - w} d\xi d\eta = 0$$

$$\Rightarrow \quad \int\int_{\Omega} \frac{H^+(\zeta)}{\zeta - w} d\xi d\eta \simeq \int\int_{\Omega/\{w\}} \frac{H^+(\zeta)}{\zeta - w} d\xi d\eta$$

- For computation of ∂H^+ and ΔH^+,

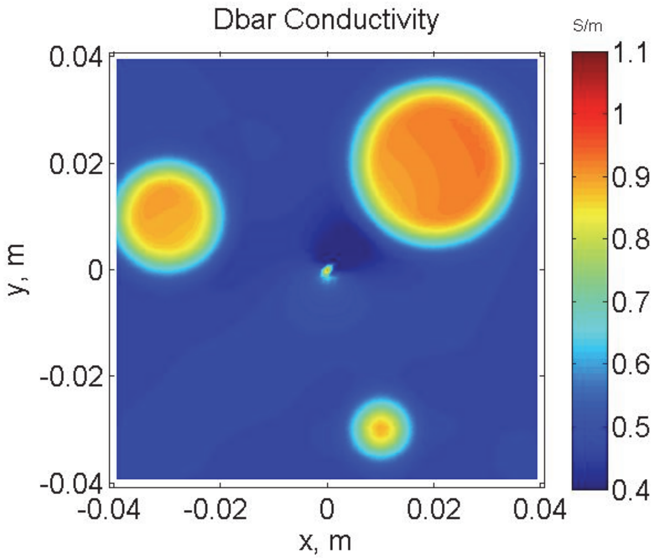
  Gaussian filter + Savitzky-Golay Filter were used.

  (2nd degree polynomial approximation)

$$\gamma(w) = \frac{4\pi i \partial H^+(w)}{\int_{\partial\Omega} \frac{\frac{2}{\gamma(\zeta)}\partial H^+(\zeta)}{\zeta - w} d\zeta + \omega\mu_0 \int\int_{\Omega} \frac{H^+(\zeta)}{\zeta - w} d\xi d\eta}$$

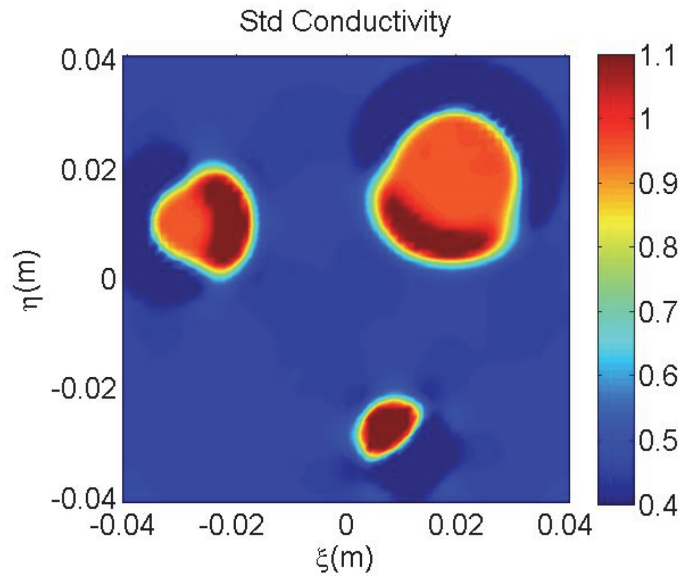
Reconstruction (noiseless)

Proposed



The three domains were well reconstructed.

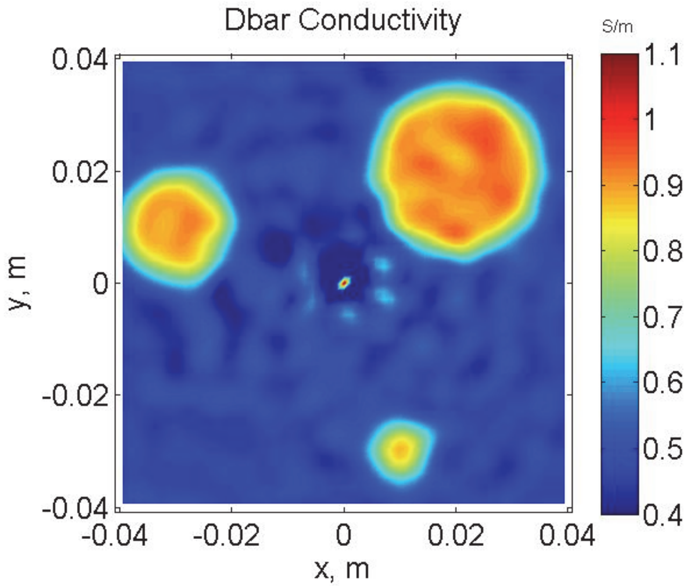
Conventional (standard)



The large errors were observed at the boundary.

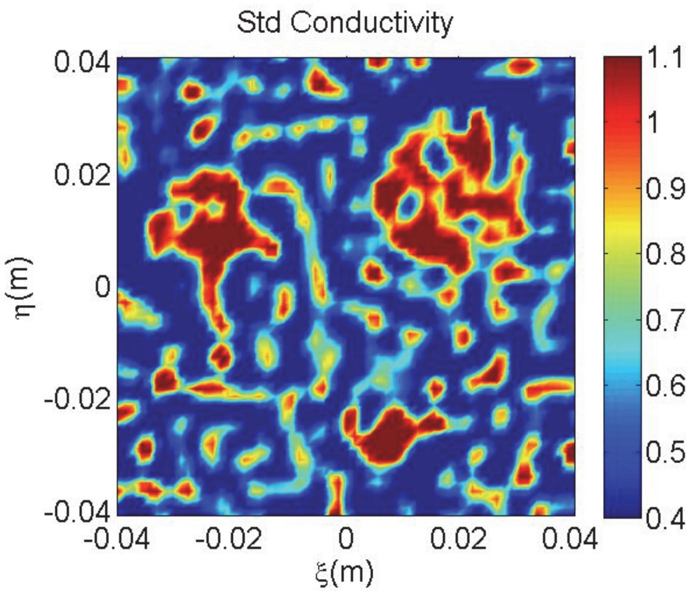
Reconstruction (1%noise)

Proposed



$$\gamma(w) = \frac{4\pi i \partial H^{+}(w)}{\int_{\Gamma} \frac{\frac{2}{\gamma(\zeta)}\partial H^{+}(\zeta)}{\zeta - w} d\zeta + \omega\mu_0 \int\int_{\Omega} \frac{H^{+}(\zeta)}{\zeta - w} d\xi d\eta}$$

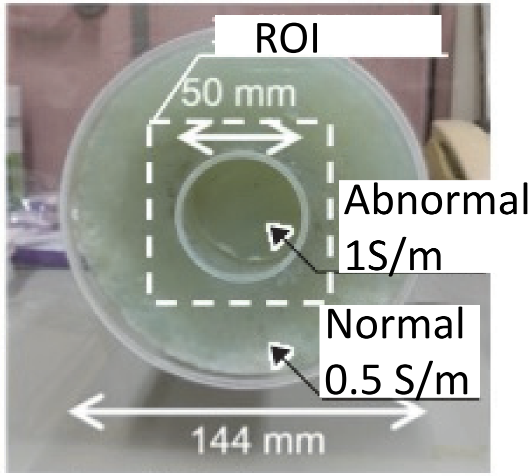
Conventional (standard)



$$\gamma = \frac{\Delta H^{+}}{i\omega\mu_0 H^{+}}$$

Phantom experiments

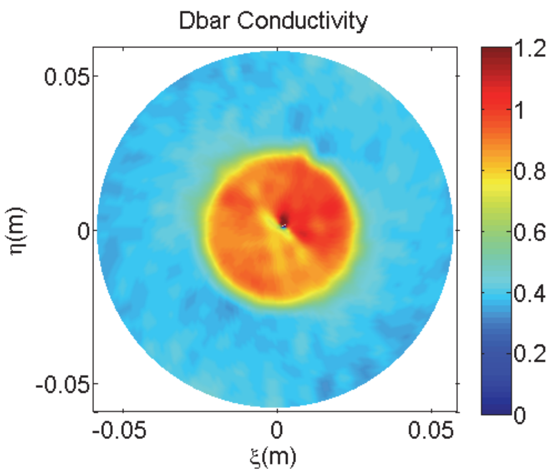

3T MRI

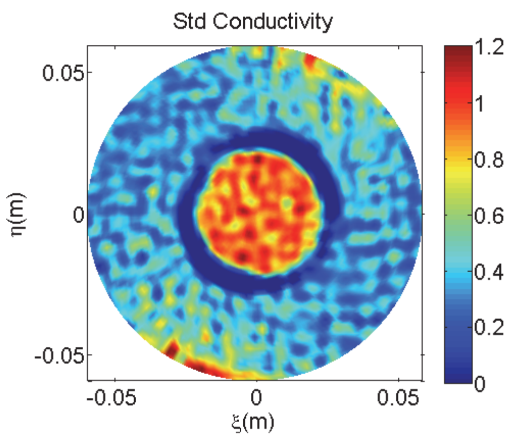


Phantom

Proposed

Conventional (standard)





- discontinuous jump of the conductivity
- robustness against noise

2. Extension to the 3D problem

Assumptions:

1) $H_z = 0$

2) $\partial_z H^+ = 0$

Usual assumptions from the structure of `birdcage coils'

~~γ does not change w.r.t the z-axis~~

- Ampere's law (z-compnents) + i $\nabla \cdot \boldsymbol{H} = 0$ $\quad 4i\partial H^+ = \gamma E_z$
- Ampere's law (xy-compnent) $\quad -\partial_z H^+ = i\gamma E^+$
- Faraday's law (xy-compnents) $\quad \bar{\partial} E_z - \partial_z E^+ = \omega\mu_0 H^+$

Extension without the assumption $\partial_z H^+ = 0$

The Dbar equation is changed to: $\bar{\partial} E_z = \omega \mu_0 H^+ + \partial_z E^+$

Iterative Algorithm:

1) Use the direct reconstruction formula to obtain an initial estimate.

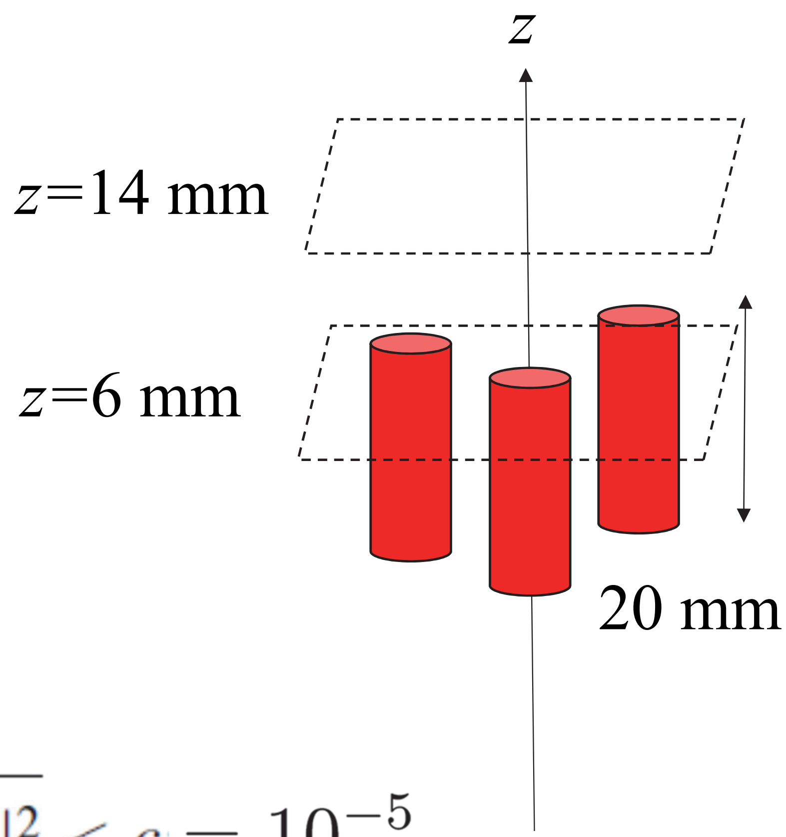
2) Compute $E^+ = \dfrac{i\partial_z H^+}{\gamma}$ with the reconstructed γ.

3) $$\gamma(w) = \frac{4\pi i \partial H^+(w)}{\int_\Gamma \frac{\frac{2}{\gamma}\partial H^+}{\zeta - w} d\zeta + \int\int_\Omega \frac{\omega\mu_0 H^+(\zeta) + \partial_z E^+(\zeta)}{\zeta - w} d\xi d\eta}$$

Go to 2).

Numerical simulations

- 64MHz, plane wave input
- Z-axis sampling: 2mm
- $\partial_z E^+$: central difference
- Gaussian filter: 3x3x3, sd=1pixel
- Convergence criteria:



$$\sqrt{\frac{1}{M}\sum_{i=1}^{M}|\gamma^{(n+1)}(\boldsymbol{r}_i)-\gamma^{(n)}(\boldsymbol{r}_i)|^2/|\gamma^{(n)}(\boldsymbol{r}_i)|^2} < c = 10^{-5}$$

Noise 1%

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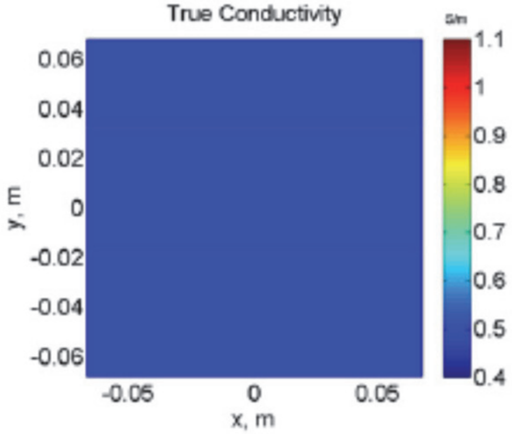
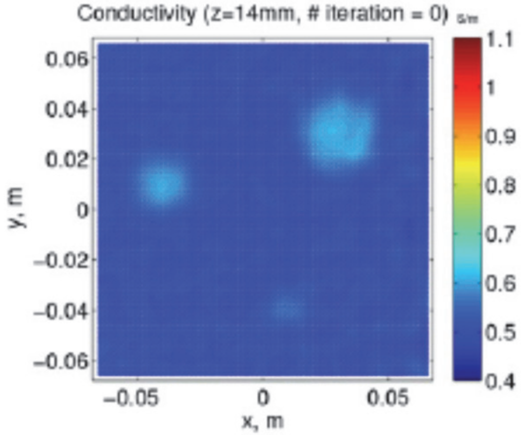
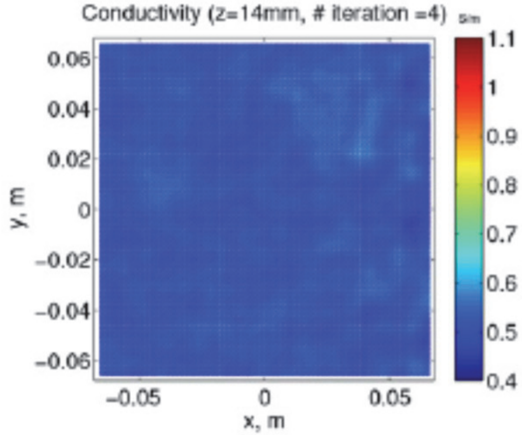
<tr><td>z=14mm</td><td>
</td><td>
</td><td>
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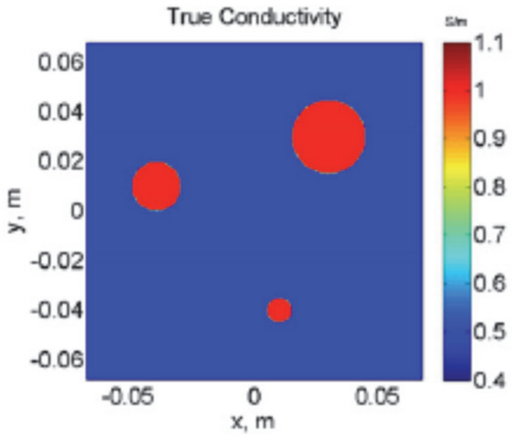
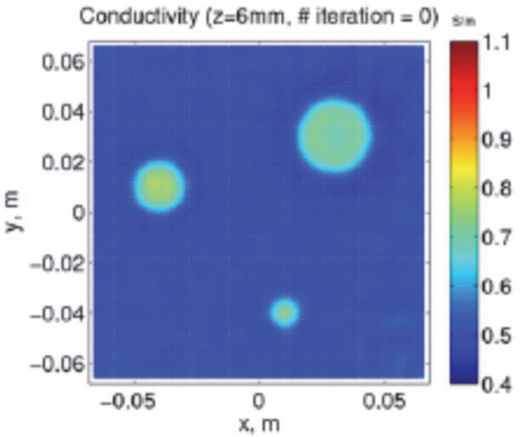
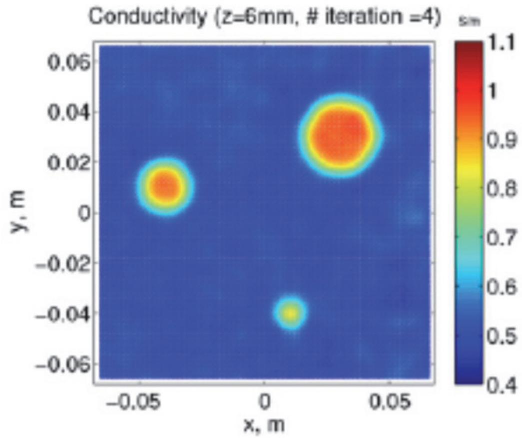
<tr><td>z=6mm</td><td>
</td><td>
</td><td>
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Concluding remarks

- Cauchy-Riemann equations: for potential fields
- Inhomogeneous Cauchy-Riemann equations: for the dual quantities in time harmonic wave fields
- E and H of the 2D time-harmonic Maxwell equations satisfy the Dbar equation, which provides a tool for computation of the unmeasurable E from the measurable H, and is thus used for reconstruction of the electric properties.
- The proposed complex-analysis-based method gives a foundation of an iterative algorithm for the 3D MREPT problem.

Takaaki Nara *et al* 2017 *Inverse Problems* **33** 105005