Complex Analysis in Industrial Inverse Problems

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How does complex analysis arise in IP?

We see complex analysis arise in two main ways:

1. **Inverse Problems** that are or can be reduced to **two dimensional** problems. Here we solve inverse problems for PDEs and integral equations using methods from complex analysis where the complex variable represents a spatial variable in the plane. This includes various kinds of tomographic methods involving weighted line integrals, and is used in medial imaging and non-destructive testing. In electromagnetic problems governed by some form of Maxwell's equation complex analysis is typically used for a plane approximation to a two dimensional problem.
2. **Frequency domain** methods in which the complex variable is the Fourier-Laplace transform variable. The Hilbert transform is ubiquitous in signal processing (everyone has one implemented in their home!) due to the analytic representation of a signal. In inverse spectral problems analyticity with respect to a spectral parameter plays an important role.

Industrial Electromagnetic Inverse Problems

Many inverse problems involve **imaging the inside from measuring on the outside**. Here are some industrial applied inverse problems in electromagnetics

- **Ground penetrating radar**, used for civil engineering eg finding burried pipes and cables. Similar also to microwave imaging (security, medicine) and radar.
- **Electrical resistivity**/polarizability **tomography**. Used to locate underground pollution plumes, salt water ingress, buried structures, minerals. Also used for industrial process monitoring (pipes, mixing vessels etc).
- **Metal detecting** and inductive imaging. Used to locate weapons on people, find land mines and unexploded ordnance, food safety, scrap metal sorting, locating reinforcing bars in concrete, non-destructive testing, archaeology, etc.

Polarization tensors I

We give a simple example first for quasi-static problems. We want to *locate and identify* an object from a perturbed field. Bounded domain $\Omega \subset \mathbb{R}^2$, $0 \in \Omega$ with conductivity or permittivity $\sigma_0 > 0$ and we consider

$$\nabla \cdot \sigma \nabla u = 0, \qquad \text{in } \mathbb{R}^2$$

with

$$u(x) - H(x) = O(|x|^{-1}) \qquad |x| \text{ as } \to \infty$$

where H is harmonic and $\sigma = \sigma_0 \chi_\Omega + \chi_{\Omega^c}$. We aim to find Ω from measurements of u outside Ω. The jump in Dirichlet and Neumann data at $\partial\Omega$ is assumed to be zero.
We use an asymptotic expansion

$$u(x) = H(x) + \sum_{|\alpha|,|\beta|=1}^{\infty} \frac{(-1)^{|\alpha|}}{\alpha!\beta!} \partial^\alpha \Gamma(x) M_{\alpha\beta}(\Omega, \sigma_0) \partial^\beta H(0)$$

Polarization tensors II

Here Γ is the Newtonian potential, and the coefficients $M_{\alpha,\beta}$ are called the **Generalized Polarization Tensor** GPT, and crucially *depend on the shape* of the object and the conductivity but the *location information in the other terms*.
To calculate the GPT of the object we need Neumann-Poincaré (NP) operator

$$K^{*}_{\partial\Omega}[\phi](x) = \mathrm{P.V.}\frac{1}{2\pi}\int\limits_{\partial\Omega} \frac{(x-x').\nu_x}{|x-x'|^2}\phi(x')\mathrm{d}x'$$

then

$$M_{\alpha\beta}(\Omega,\sigma_0) = \int\limits_{\partial\Omega} y^{\alpha}(\lambda - K^{*}_{\partial\Omega})^{-1}\left[\nu_y \cdot y^{\beta}\right]\mathrm{d}y$$

where $\lambda = (\sigma_0+1)/(2(\sigma_0-1))$
For details see Ammari & Kang [1]

More about PTs

- For the lowest order term can be considered as a symmetric matrix, the Pólya–Szegö tensor. Known explicitly for ellipses[1], but not even triangles! Eigenvalues used to distinguish between objects
- Possibly weakly electric fish 'know about it'. [7].
- The GPT expansion is also known for other electromagnetic problems including metal detecting [9, 8, 10]. Acoustics [1]. Full Maxwell's [14] hence radar. GPTs are can be thought of as an asymptotic expansion for radar cross section.

But what are recent uses of complex analysis? ....

Recent developments using (old) complex analysis

Choi et al [4] made progress on explicit calculation of the GPT. They start with an asymptotic expansion of the conformal map from the exterior of a disk to the exterior of Ω. They write this mapping Ψ as an asymptotic series

$$\Psi(w) = w + a_0 + \frac{a_1}{w} + \frac{a_2}{w^2} + \cdots$$

and note the a_n can be solved from operator equation using the NP operator. They go on to define the Faber polynomials $\{F_m(z)\}$ associated with Ψ which are complex polynomials and form a basis for complex analytic functions in Ω. If $z = \Psi(w)$ then

$$\log(z - \tilde{z}) = \log(w) - \sum_{m=1}^{\infty} \frac{1}{m} F_m(\tilde{z}) w^{-m}$$

with a suitable branch cut. They go on to define GPT in terms of these Faber polynomials. The trick is that the components of these Faber polarization tensors can be computed in terms of the a_n. They also use the method to create *neutral inclusions*, which are in a sense *cloaked* to a low order.

A spectral example I

In the magnetic induction case we hope to use the frequency dependence of the polarization tensor to better distinguish between objects. We studied the spectral properties in [10].

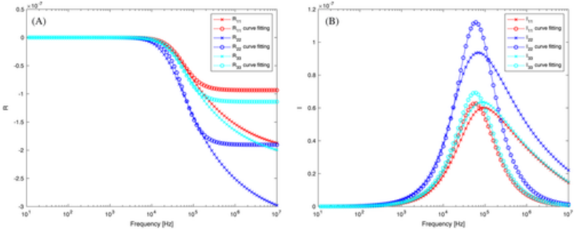

Figure show real and imaginary part of magnetic polarization tensor (MPT) as a function of frequency.

A spectral example II

We wanted to understand the eigenvalues of the real and imaginary parts of the MPT and if the “sigmoid and hump” we observed could be explained.
We found that $M(w)$ where imaginary w is frequency is meromorphic with poles λ_i on the positive real axis, the eigenvalues of a certain curl-curl operator involving the domain, and admits a **Mittag-Leffler** type expansion that goes some way to explaining the behaviour.
It also provides a connection between the inverse problem of finding the shape using metal detector data at a range of frequencies and inverse spectral theory for operators.

Attenuated Radon transform I

The inversion and uniqueness theory of the attenuated Radon transform in the plane uses complex analysis. This is at the heart of **emission computed tomography** methods such as **SPECT** used in medical imaging, and the techniques are increasingly relevant as new methods are developed for tomographic imaging of industrial as well as medical problems. We will give a flavour.

The divergent beam transform

$$Da(x,\theta) = \int_0^\infty a(x+s\theta)\,\mathrm{d}s, \quad x \in \mathbb{R}^2, \theta \in S^1$$

The attenuated x-ray (or attenuated Radon) transform [2, 12]

$$P_a f(x,\theta) = \int_{\mathbb{R}} \exp(-Da(x+s\theta,\theta)) f(x+s\theta)\,\mathrm{d}s$$

In applications f is the **density of emitters** while a is the **attenuation**. For a known we seek f from $P_a f$.

Attenuated Radon transform II

We can reformulate the integral operator as a partial differential operator thought of as a *transport equation* for a "flux" ψ

$$\theta\partial_x \psi(x, \theta) + a(x)\psi(x, \theta) = f(x)$$

where $\theta\partial_x = \theta_1\partial_1 + \theta_2\partial_2$ and we consider $\psi^+(x, \theta)$ as the solution with the initial condition

$$\lim_{s\to-\infty} \psi^+(x + s\theta, \theta) = 0$$

so that

$$P_a f(x, \theta) = \lim_{s\to\infty} \psi^+(x + s\theta, \theta)$$

Novikov [12] proved the following uniqueness and explicit inversion result.

Novikov's theorem

See Theorem 2.1 of Novikov [12].
http://dx.doi.org/10.1007/BF02384507.

Note Novikov's use of the Hilbert transform, as well as the standard plane Radon transform, and the appearance of the $\partial/\partial z$ operator, see also Fokas and Novikov [5]. See [3, 2] for treatment of the attenuated Radon transform as a Cauchy problem for *A-analytic* functions....

A-analytic functions (Bukhgeim)

See Bukhgeim and Bukhgeim[3] p220
`http://dx.doi.org/0.1515/156939406777340883`

Further uses

These ideas from the scalar attenuated Radon transform have been extended to the tomography of vector fields which arises in Doppler velocimetry and in neutron strain tomography, see [6], [11].
For the attenuated ray transform in a geometric context on a Reimannian manifold see for example Salo and Uhlmann [13].
In radio propagation through a non-uniform medium (eg ionosphere, ground) linearization results in weighted ray transforms, hence applications to radar. Also polarized neutron tomography of magnetic fields.

Conclusions

There are plenty of areas of inverse problems used in industry where a knowledge of complex analysis is needed, but it is not clear that it is really generating new problems in complex analysis.

Thanks for listening.

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