Term Structure Models

Mike Tehranchi
Thomas Du Toit

CCIMI Cantab Capital Institute For the Mathematics Of Information
University of Cambridge

March 19, 2019

Outline

Term Structure Models

Term Structure Models

Term Structure Models = Models for interest rates $(r_t)_{t\geq 0}$

Term Structure Models

Term Structure Models = Models for interest rates $(r_t)_{t\geq 0}$
We assume we are in a market with $d+1$ assets where we have a risk less asset called the bank account given by

$$B_t = B_0 e^{\int_0^t r_s ds}$$

Term Structure Models

Term Structure Models = Models for interest rates $(r_t)_{t\geq 0}$
We assume we are in a market with $d+1$ assets where we have a risk less asset called the bank account given by

$$B_t = B_0 e^{\int_0^t r_s ds}$$

and a $\mathbb{R}^d$-valued RCLL Semi-martingale $(S_t)_{t\geq 0}$.

Term Structure Models

Term Structure Models = Models for interest rates $(r_t)_{t\geq 0}$
We assume we are in a market with $d+1$ assets where we have a risk less asset called the bank account given by

$$B_t = B_0 e^{\int_0^t r_s ds}$$

and a $\mathbb{R}^d$-valued RCLL Semi-martingale $(S_t)_{t\geq 0}$.
From now on we will denote (B_t, S_t) by P_t.

What is Arbitrage?

What is Arbitrage?

According to the Cambridge dictionary: Arbitrage is the method on the stock exchange of buying something in one place and selling it in another place at the same time, in order to make a profit from the difference in price in the two places.

What is Arbitrage?

According to the Cambridge dictionary: Arbitrage is the method on the stock exchange of buying something in one place and selling it in another place at the same time, in order to make a profit from the difference in price in the two places.
We use a more general definition and get in the language of mathematics the following:

Arbitrage

An (absolute) arbitrage opportunity is a self-financing predictable $\mathbb{R}^{d+1}$-valued process H such that

$$H_0 \cdot P_0 \leq H_T \cdot P_T \quad a.s.$$
$$\mathbb{P}[H_T \cdot P_T > 0] > 0$$

for some $T > 0$.

1. Fundamental Theorem of Asset Pricing

1. Fundamental Theorem of Asset Pricing

A market is a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ (where $\mathbb{F}$ satisfies the usual conditions) with a semi martingale P describing the asset prices.

A market is a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ (where $\mathbb{F}$ satisfies the usual conditions) with a semi martingale P describing the asset prices.

1. Fundamental Theorem of Asset Pricing

If there exist an equivalent probability measure $\mathbb{Q}$ such that the process $\frac{S}{B}$ is a $\mathbb{Q}$-local martingale then there is no arbitrage.

Model Set-up

We assume we have a $\mathbb{R}^N$-valued process Z given by

$$dZ_t = \beta(Z_t)dt + \sigma(Z_t)dW_t$$

called the economic factor where $\beta : \mathbb{R}^N \to \mathbb{R}^N$, $\sigma : \mathbb{R}^N \to \mathbb{R}^{N\times m}$ and W is a m-dimensional $\mathbb{Q}$ - Brownian motion.

Model Set-up

We assume we have a $\mathbb{R}^N$-valued process Z given by

$$dZ_t = \beta(Z_t)dt + \sigma(Z_t)dW_t$$

called the economic factor where $\beta : \mathbb{R}^N \to \mathbb{R}^N$, $\sigma : \mathbb{R}^N \to \mathbb{R}^{N\times m}$ and W is a m-dimensional $\mathbb{Q}$ - Brownian motion. Further a function $R : \mathbb{R}^N \to \mathbb{R}$ is given such that

$$r_t = R(Z_t)$$

Model Set-up

We assume we have a $\mathbb{R}^N$-valued process Z given by

$$dZ_t = \beta(Z_t)dt + \sigma(Z_t)dW_t$$

called the economic factor where $\beta : \mathbb{R}^N \to \mathbb{R}^N$, $\sigma : \mathbb{R}^N \to \mathbb{R}^{N\times m}$ and W is a m-dimensional $\mathbb{Q}$ - Brownian motion. Further a function $R : \mathbb{R}^N \to \mathbb{R}$ is given such that

$$r_t = R(Z_t)$$

and a bank process given by

$$B_t = B_0 e^{\int_0^t r_s ds}$$

Arbitrage-free pricing

Arbitrage-free pricing

We are now interested in introducing a European claim into the market.

Arbitrage-free pricing

We are now interested in introducing a European claim into the market.

European claim

Fix a time $T > 0$. An European claim with maturity T is a $\mathcal{F}_T$ -measurable random variable C.

Arbitrage-free pricing

By the Feynman-Kac Theorem we know

Feynman-Kac

Let $V : [0, T] \times \mathbb{R}^N \to \mathbb{R}$ be a C^2 function which satisfies the following PDE

$$\frac{\partial V}{\partial t} + \sum_{i}^{N} \beta^i \frac{\partial V}{\partial z^i} + \frac{1}{2} \sum_{i,j}^{N} \alpha^{i,j} \frac{\partial^2 V}{\partial z^i z^j} = rV$$

with terminal condition

$$V(T, z) = g(z)$$

where $g : \mathbb{R}^N \to \mathbb{R}_{\geq 0}$ and $\alpha^{i,j} = (\sigma \sigma^{Tr})_{i,j}$. Then $\frac{V}{B}$ is a $\mathbb{Q}$ - local martingale and thus we have no arbitrage.

Research Questions

Research Questions

We want to price zero-coupon bonds for the case $N = 1$.

Research Questions

We want to price zero-coupon bonds for the case $N = 1$. Does the price process given by

$$V(t,z) = \sum_{k=0}^{n} g_k(t) z^k$$

with

$$V(T,z) = 1$$

satisfy the Feynman Kac Formula? and if so what are the constraints?

Extension of the Model

We now assume the economic factor has the following dynamics

$$dZ_t = \beta(Z_t)dt + \sigma(Z_t)dW_t + \gamma(Z_t)dN_t$$

where N is a Poisson process with intensity λ. Does the price process given by

$$V(t,z) = \sum_{k=0}^{n} g_k(t)z^k$$

with

$$V(T,z) = 1$$

satisfy the Feynman Kac Formula? If so what are the constraints? What about if we use a Levy process?.