Sequential testing and high-dimensional online change point detection

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Setting

- Change point detection
  $X_1, \ldots, X_{z-1} \overset{\text{iid}}{\sim} N_p(0, \sigma^2 I_p)$, $X_z, X_{z+1}, \ldots \overset{\text{iid}}{\sim} N_p(\theta, \sigma^2 I_p)$
- (One-sided) sequential testing $X_1, X_2, \ldots \overset{\text{iid}}{\sim} N_p(\theta, \sigma^2 I_p)$
  $H_0 : \theta = 0 \quad vs. \quad H_1 : \theta \neq 0$
  One-sided: never accepts H_0
- Assume σ^2 known, z, θ unknown
- Minimal signal strength and sparsity assumption on θ

Change point detection: offline vs. online

- Offline: fixed sample size n, one or multiple mean shifts occur within $X_1, \ldots, X_n$
- CUSUM statistics (e.g. Wang and Samworth, 2018)

Change point detection: offline vs. online

- Offline: fixed sample size n, one or multiple mean shifts occur within $X_1, \ldots, X_n$
- CUSUM statistics (e.g. Wang and Samworth, 2018)
- Sequential: Observations $X_1, X_2, \ldots$ come in one at a time
- Declare change as quickly as possible after it takes place
- Online: In addition, computational complexity at each step does NOT depend on the current sample size

Quantities of interest

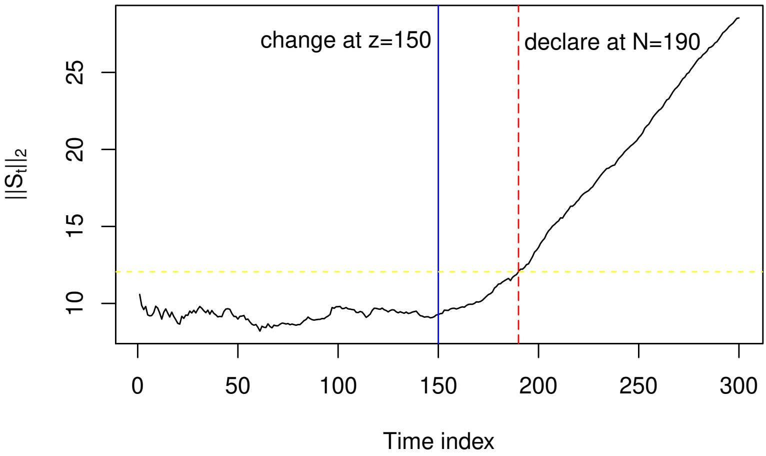
- Stopping time N, possibly ∞
- 'Patience' of the procedure: measured by the average run length till a false alarm under the null $E_0(N)$
- 'Alertness' of the procedure: measured by the average run length from change to reaction if a change is present, given the most adversarial null sequence before the change

$$\bar{E}_\theta(N) = \sup_{z \in \mathbb{N}} \operatorname{ess\,sup} E_{z,\theta}[N - z + 1 | X_1, \ldots, X_{z-1}]$$

- Goal:

$$\min_{N: E_0(N) \geq \gamma} \bar{E}_\theta(N)$$

Quantities of interest



Online change point detection vs. sequential testing

- Lorden (1971) suggested that we can always build a sequential change point detection procedure with theoretical guarantees from an appropriate one-sided sequential testing procedure;
- It is NOT clear that in general this approach will give us online procedures (computational complexity consideration);
- In univariate exponential family case, we can construct an online change point detection algorithm according to Page (1954) and Lorden (1971) for composite alternative.

One-sided sequential testing

(One-sided) sequential testing $X_1, X_2, \ldots \overset{\text{iid}}{\sim} N_p(\theta, \sigma^2 I_p)$
$H_0 : \theta = 0 \quad vs. \quad H_1 : \theta \neq 0 \qquad$ (never accepts H_0)

Assume $\sigma^2 = 1$. Denote $S_t := X_1 + \ldots + X_t$. Let

$$N := \inf\{t \in \mathbb{N} : t \geq 5, \|S_t\|_2^2/t \geq p + \sqrt{p}\xi_t\},$$

where $\xi_t = C\left(\sqrt{\log(1/\epsilon)\log\log t} + \frac{\log(1/\epsilon)\log\log t}{\sqrt{p}}\right)$ for some large enough constant $C > 0$.

One-sided sequential testing

- Probability of ever stopping under the null: $P_0(N < \infty) \leq \epsilon$
- Upper bound (up to a log-log factor) on the average run length under the alternative:

$$E_\theta(N) \lesssim \frac{\sqrt{p}}{\|\theta\|_2^2} \left(\sqrt{\log(1/\epsilon)} + \frac{\log(1/\epsilon)}{\sqrt{p}} \right)$$

- Working on the corresponding lower bound
- $\longrightarrow$ sequential change point procedure

One-sided sequential testing

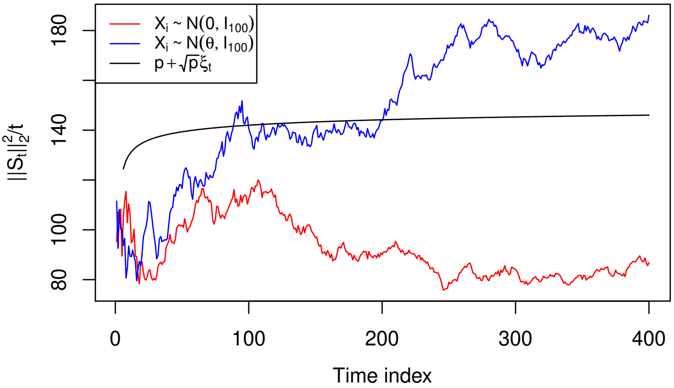


Figure: Sequential testing procedure with $p = 100, \epsilon = 1/8$ and $\vartheta = 1/2$

Future work

- Include sparsity in high dimensions
- Use sequential testing or otherwise to construct a change point detection procedure
- Spatial dependence between co-ordinates of the data stream

Thank you for listening!

Lorden, G. (1971) Procedures for reacting to a change in distribution. *Ann. Math. Statist.*, **42**, 1897–1908.

Page, E. S. (1954) Continuous inspection schemes. *Biometrika*, **41**, 100–115.

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