Mechanics from Continuous and Localised Intrinsic Curvature Creation

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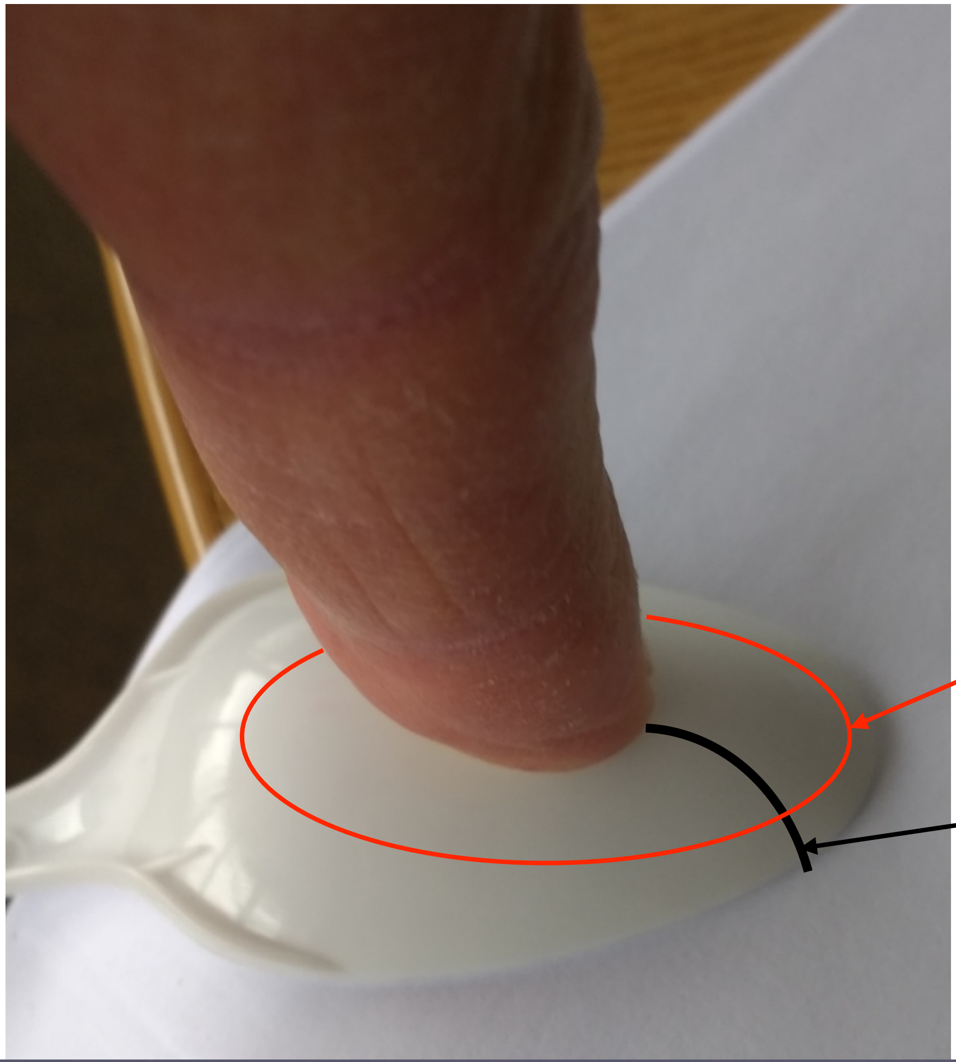
Why curvature from flat sheets?

• Strong actuation if developing curvature is blocked.

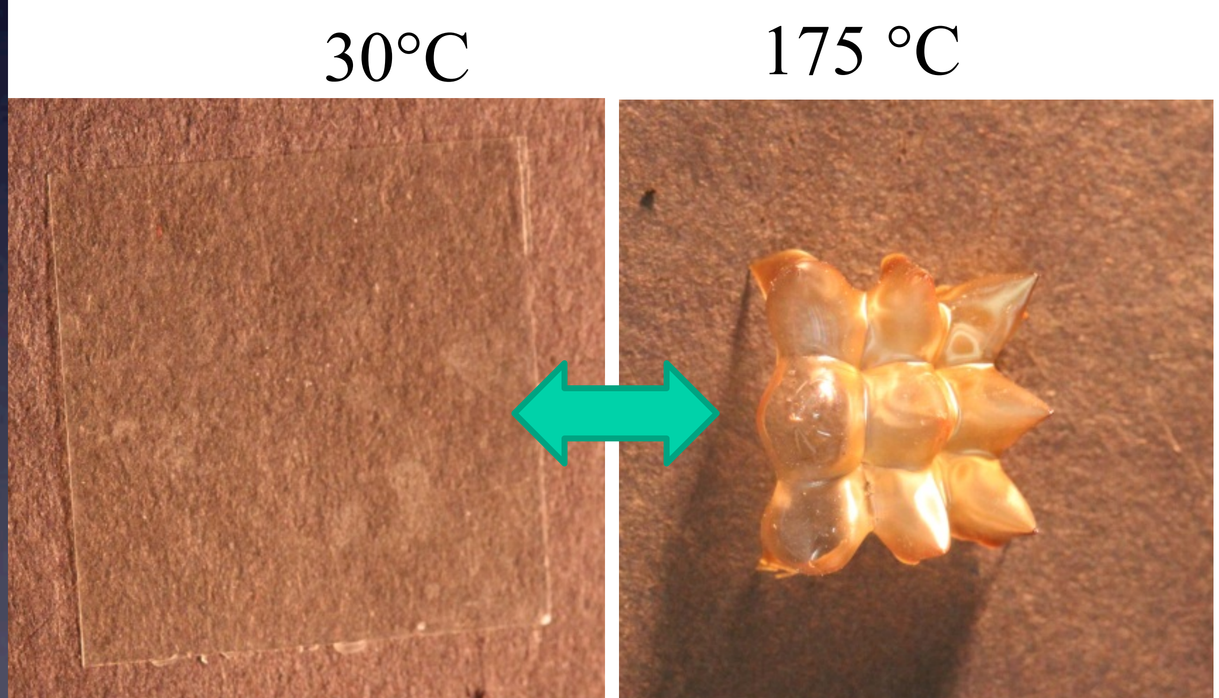
• Induce stretch & compression (not just bend).



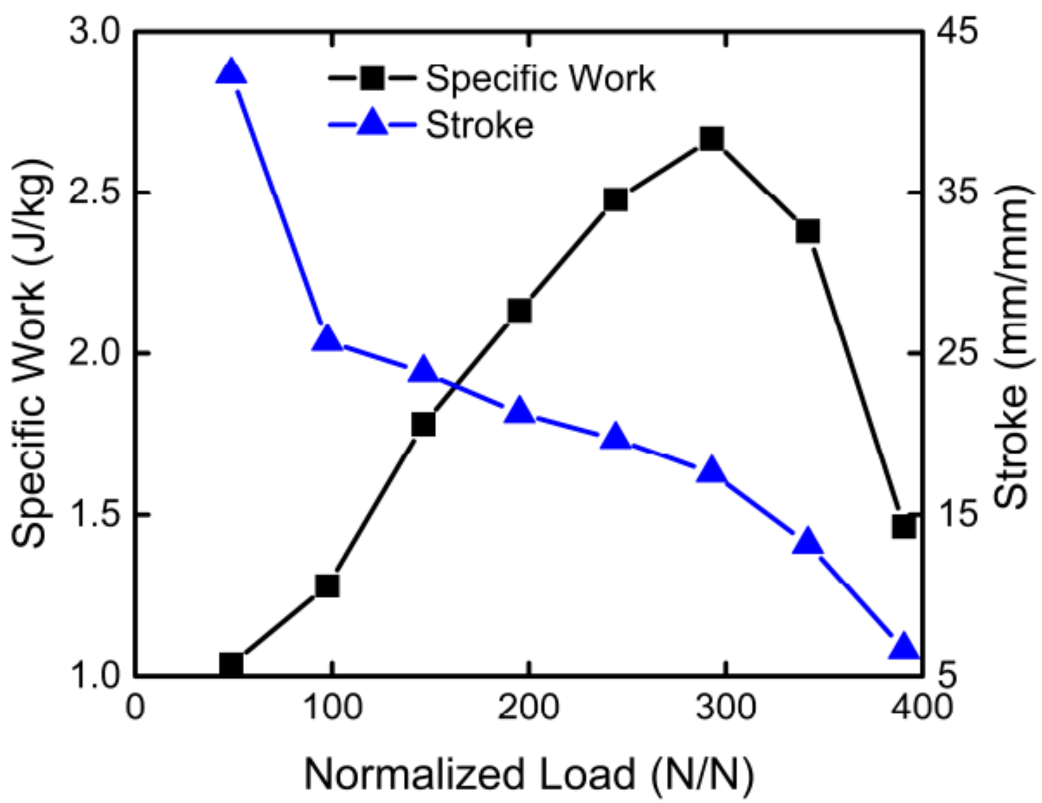
Bend of 2-D curved object difficult

- cup of spoon
- imagine it were developing against a load.

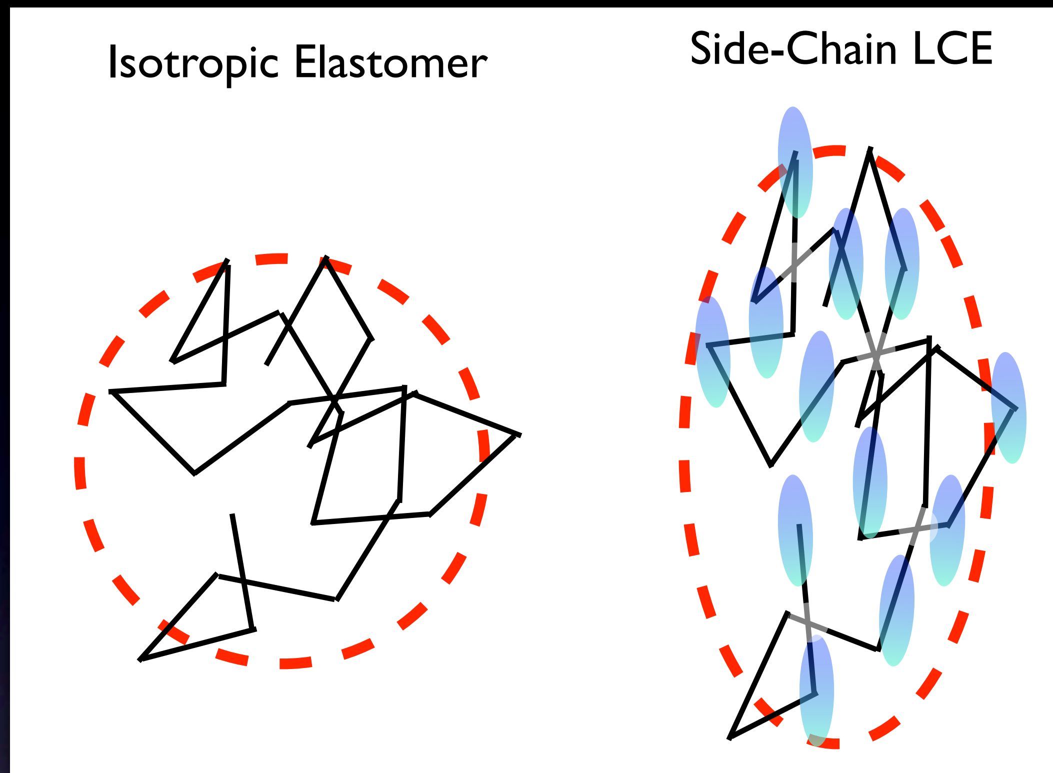
- Reversible, shape-changing materials.
- Imprint designs.
- Steerable with light, heat, . .

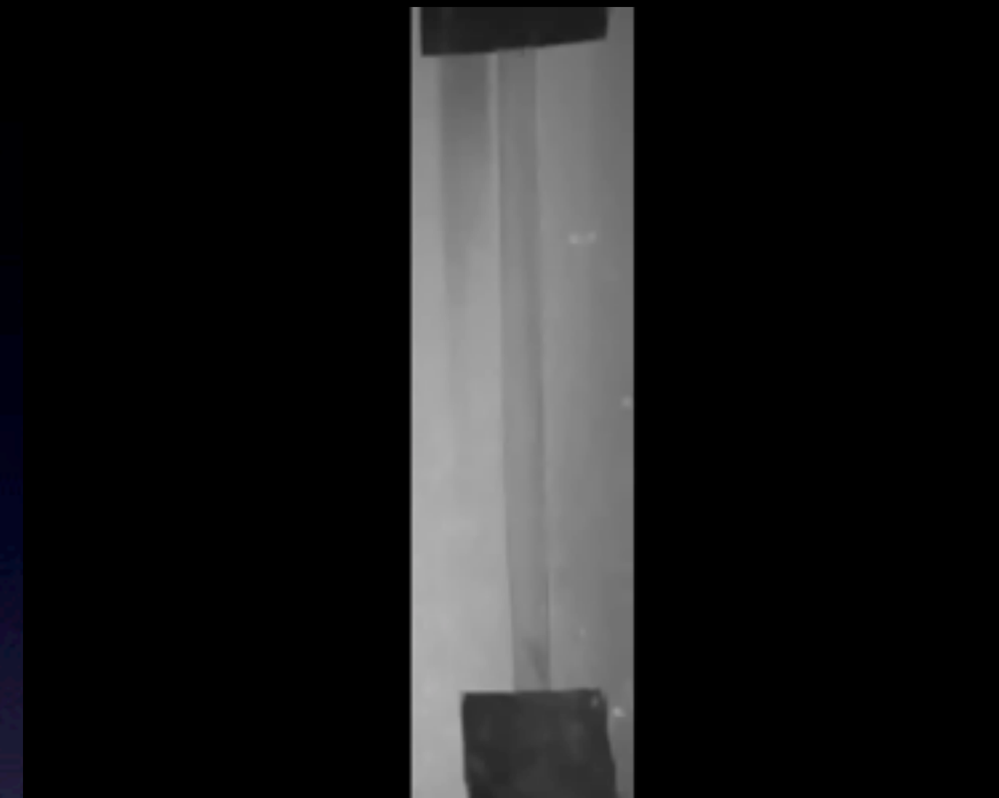


(Tim White *et al*)



Liquid Crystal Solids





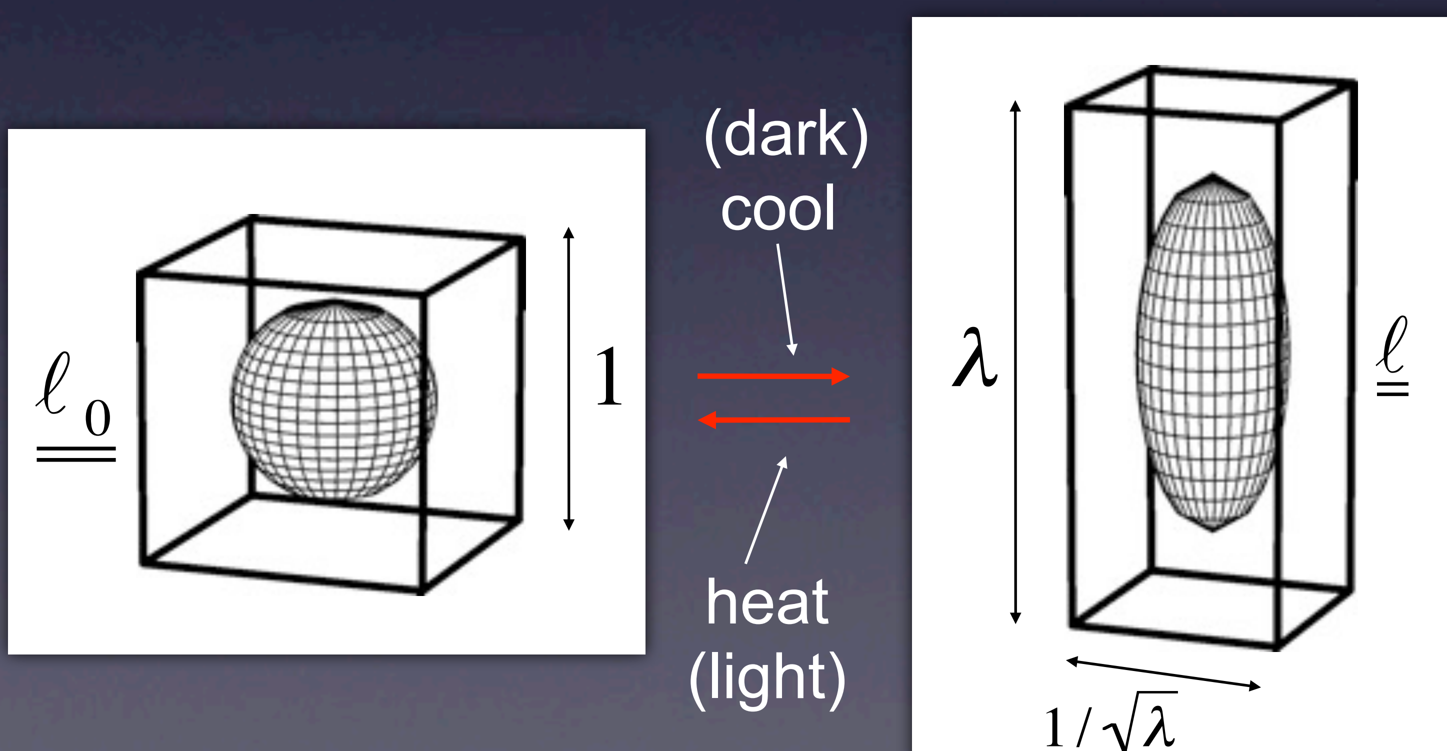
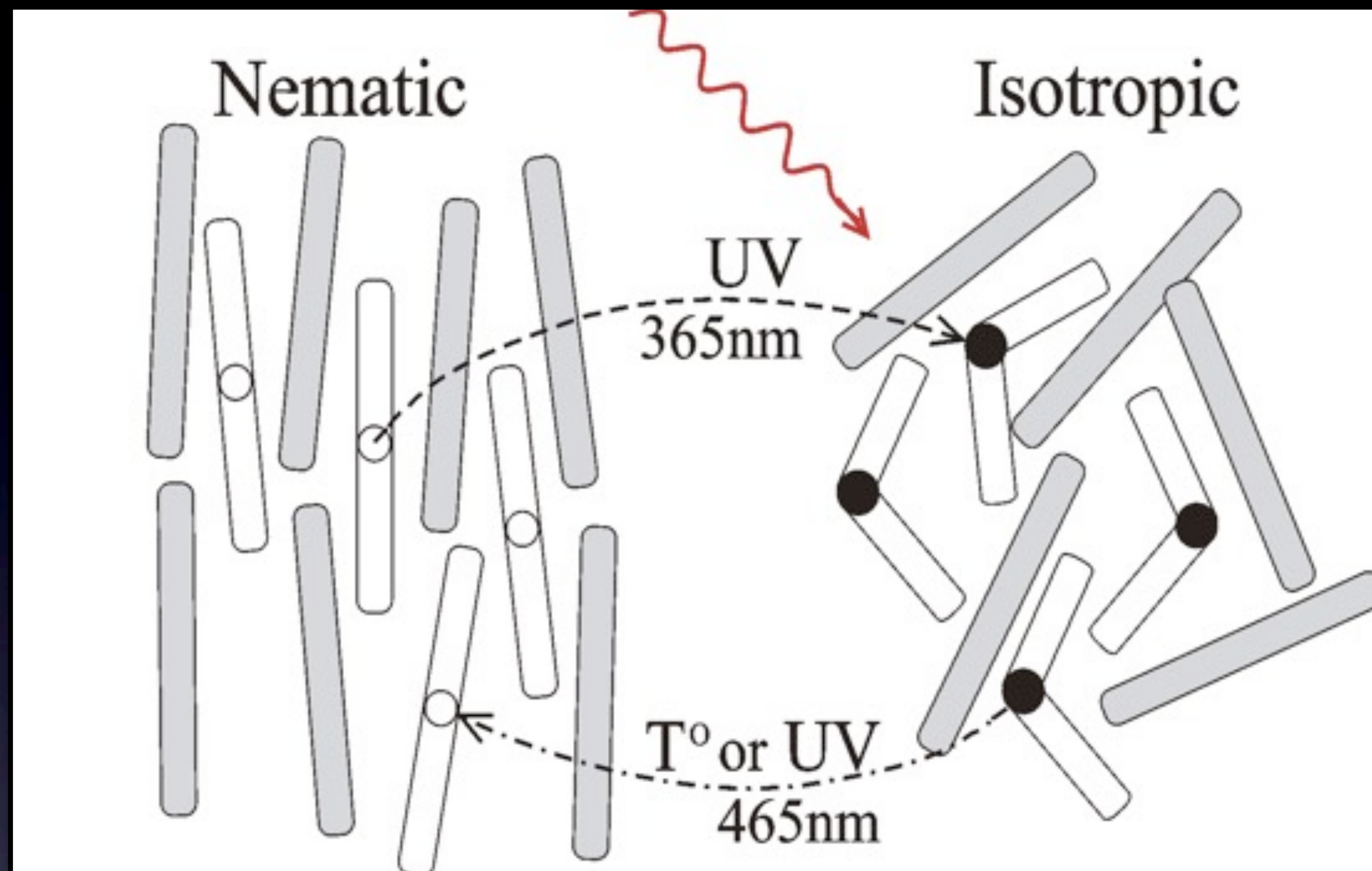
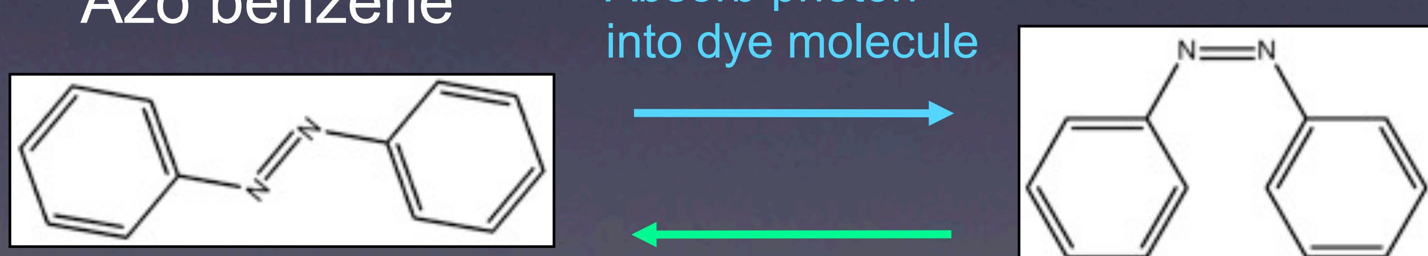


Photo-Actuation as an Achievable Option

- Photo alternative to thermal disruption of order.



Azo benzene

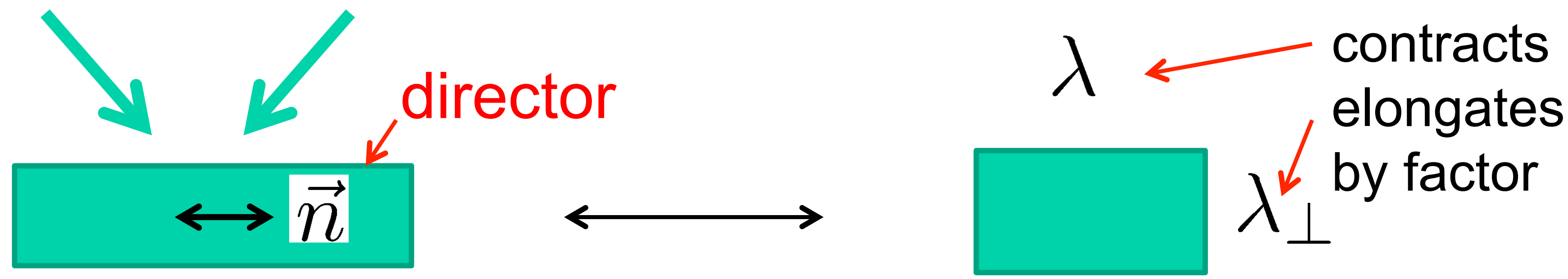


trans isomer
(straight)

cis isomer
(bent)

Liquid Crystal Rubber & Glass

Heat and Light-induced, new, relaxed state:



	cool	heat		cool	heat	
Parallel	$\lambda \sim 1.1$	$- \; 0.9$	(glass)	~ 4	$- \; 0.25$	(rubber)

Liquid Crystal Rubber & Glass

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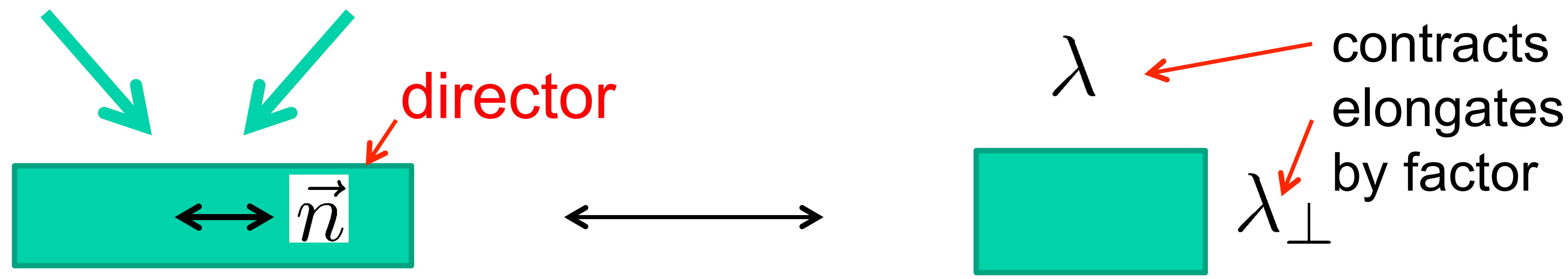

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opto-thermal Poisson ratio $\nu \sim 0.5 \longleftrightarrow 2$ glass

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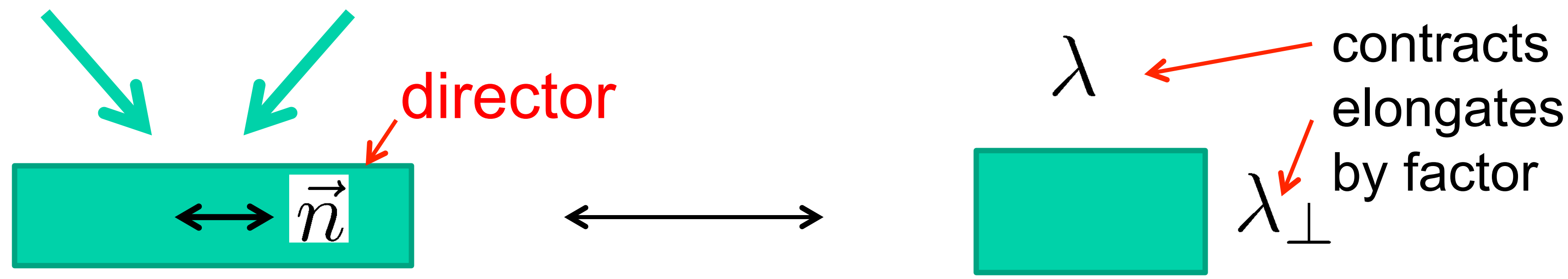

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$$\underline{\underline{\lambda}} = (\lambda - \lambda^{-\nu})\vec{n}\,\vec{n} + \lambda^{-\nu}\underline{\underline{\delta}}$$

deformation gradient tensor

Topography from in-plane variation of $\underline{\underline{\lambda}}$

via director variation: $\vec{n}(\vec{r})$

Magnitude λ spatially constant, varies with T and light.

Aim: Gaussian-curved surfaces of revolution.

Aharoni *et al* (2014) – Cartesian patterns of $\vec{n}(\vec{r})$, also the inverse problem. Sharon, Efrati *et al*. (2010, 2007 – review, swelling, circular + Cartesian, leaves, plates, . .)

Mostajeran *et al* (2015, 2016, 2018) – circularly symmetric director, anchoring, inverse problem, evolution of directors and material curves.

Lengths

space

basis (flat)

$$d\vec{R}^2 = dR_i dR_i \equiv \lambda_{il} dR_{\mathrm{o}l} \lambda_{ik} dR_{\mathrm{o}k}$$

$$= dR_{\mathrm{o}l} \underline{\lambda^{\mathrm{T}}_{li} \lambda_{ik}} dR_{\mathrm{o}k} \equiv dR_{\mathrm{o}l} \underline{a_{lk}} dR_{\mathrm{o}k}$$

Metric tensor of current, relaxed state.

Metric

$$\underline{\underline{a}} = (\lambda^2 - \lambda^{-2\nu})\vec{n}\,\vec{n} + \lambda^{-2\nu}\underline{\underline{\delta}}$$

Spatial variation gives Gaussian Curvature.

Metric Programming

$$\underline{\underline{\lambda}} = (\lambda - \lambda^{-\nu})\underline{n}\,\underline{n} + \lambda^{-\nu}\underline{\underline{\delta}} \qquad\qquad \underline{\underline{g}} = \underline{\underline{\lambda}}^{T}\underline{\underline{\lambda}}$$

In polar coordinates the metric then becomes:

$$\begin{pmatrix} \lambda^2\cos^2(\phi-\theta) + \lambda^{-2\nu}\sin^2(\phi-\theta) & (\lambda^2 - \lambda^{-2\nu})\sin(\phi-\theta)\cos(\phi-\theta) \\ (\lambda^2 - \lambda^{-2\nu})\sin(\phi-\theta)\cos(\phi-\theta) & \lambda^2\sin^2(\phi-\theta) + \lambda^{-2\nu}\cos^2(\phi-\theta) \end{pmatrix}$$

$$\Gamma^i_{jk} = \tfrac{1}{2}\sum_l g^{il}(\partial_j g_{lk} + \partial_k g_{jl} - \partial_l g_{jk})$$

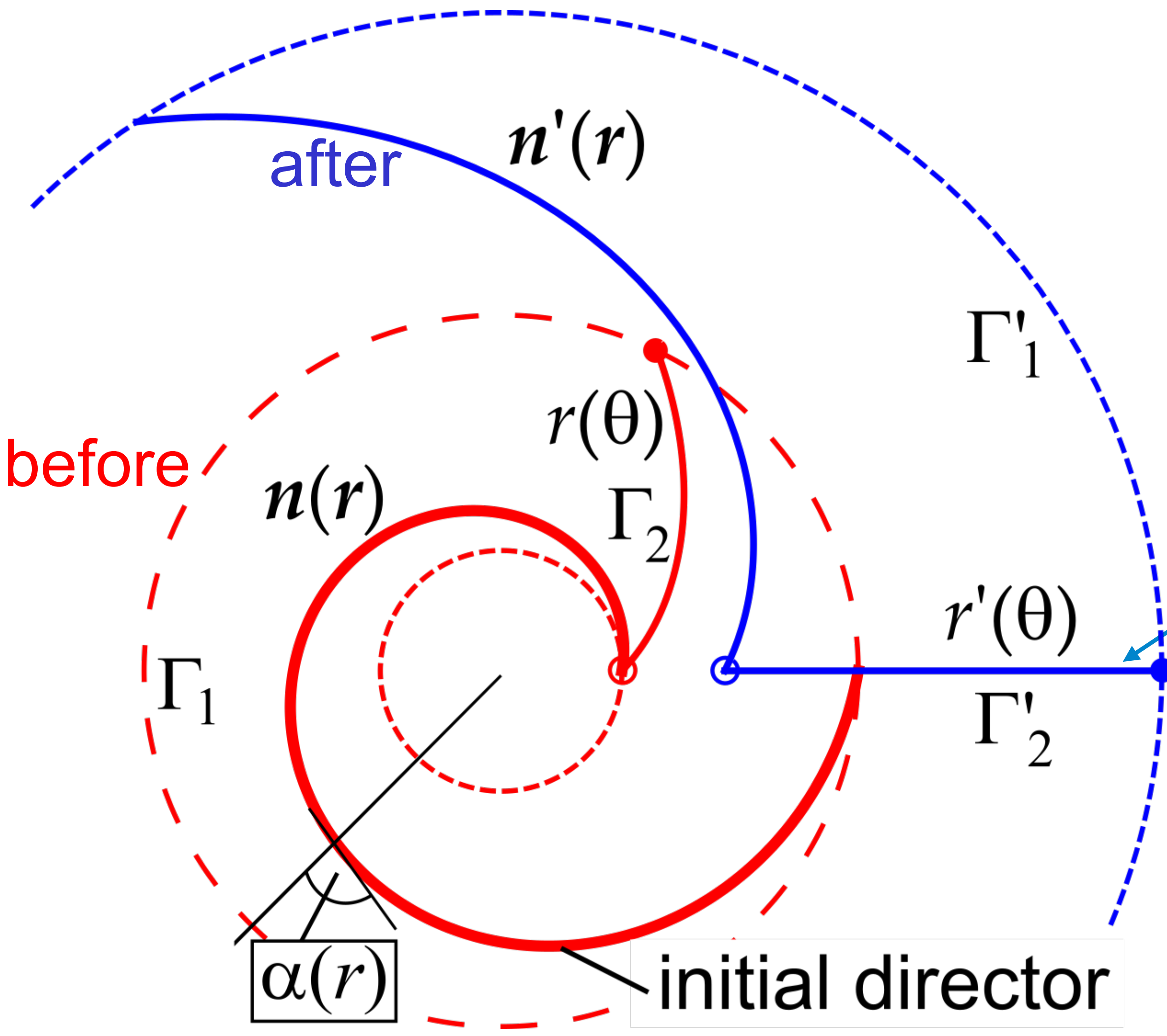
$$K = -\frac{1}{g_{rr}}\left[\Gamma^{\theta}_{r\theta,r} - \Gamma^{\theta}_{rr,\theta} + \Gamma^{r}_{r\theta}\Gamma^{\theta}_{rr} - \Gamma^{r}_{rr}\Gamma^{\theta}_{r\theta} + (\Gamma^{\theta}_{r\theta})^2 - \Gamma^{\theta}_{rr}\Gamma^{\theta}_{\theta\theta}\right]$$

Substituting the form of a Frank minimizing texture for a single defect, for example, yields:

$$\boxed{K = \frac{m(m-1)\left(\lambda^{2(1+\nu)} - 1\right)}{r^2\lambda^2}\cos\left[2(m-1)\theta\right]}$$

C.D. Modes and M. Warner, *Phys. Rev. E* 92(1) 010401 (2015)

Example of director variation $\vec{n}(\vec{r})$
Circularly symmetric – variation in $\alpha(r)$, angle of $\vec{n}$ to radial.



Evolution with λ of
- director spirals $\vec{n}(\vec{r})$
- circles Γ_1
- proto-radii Γ_2

(in material)

Challenges

- Forward: given $\alpha(r)$, what shells result as vary λ
- Inverse: given shell shape and λ, what $\alpha(r)$

$\underline{\underline{a}}(\vec{r})$ does not directly give shell – need to construct surface.

Also questions of embedding, circular symmetry, anchoring.

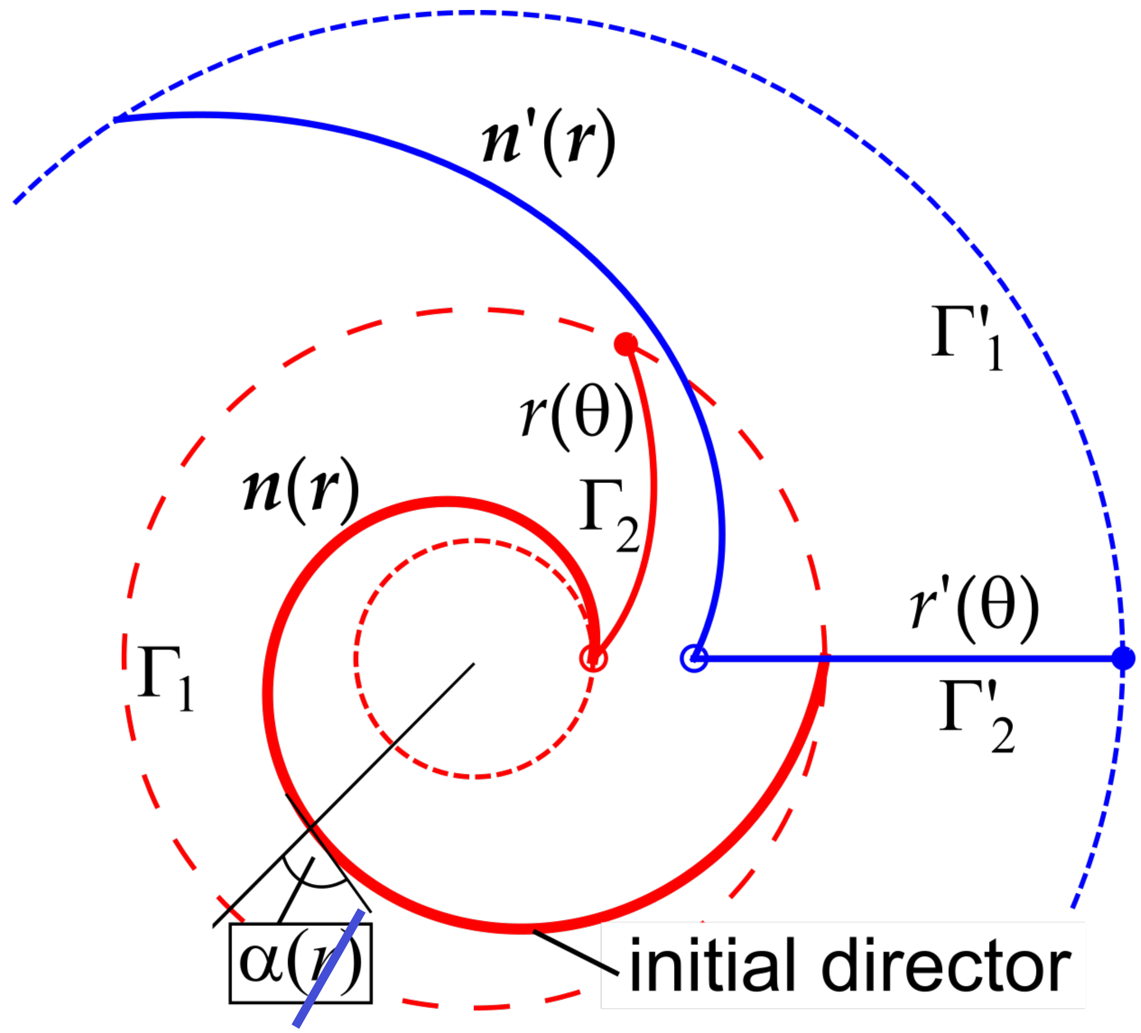
Mostajeran, MW

Cone – simplest – already subtle, even though GC = 0.

$\alpha = \text{const.}$ director curves are log spirals

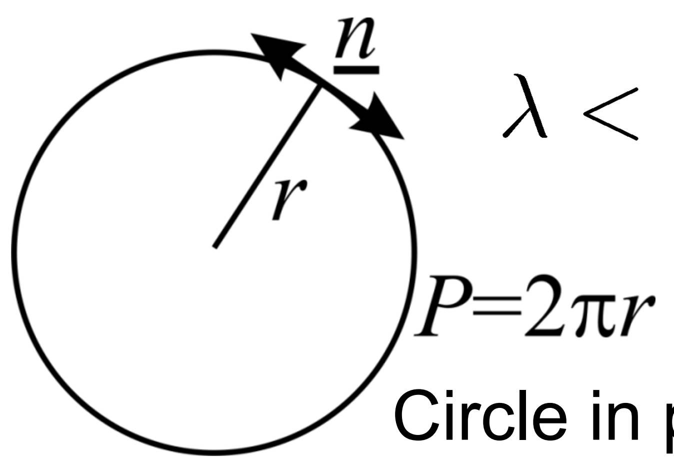
$K(r) = 0$ (except at tip)

Modes, Bhattacharya, MW (2010)

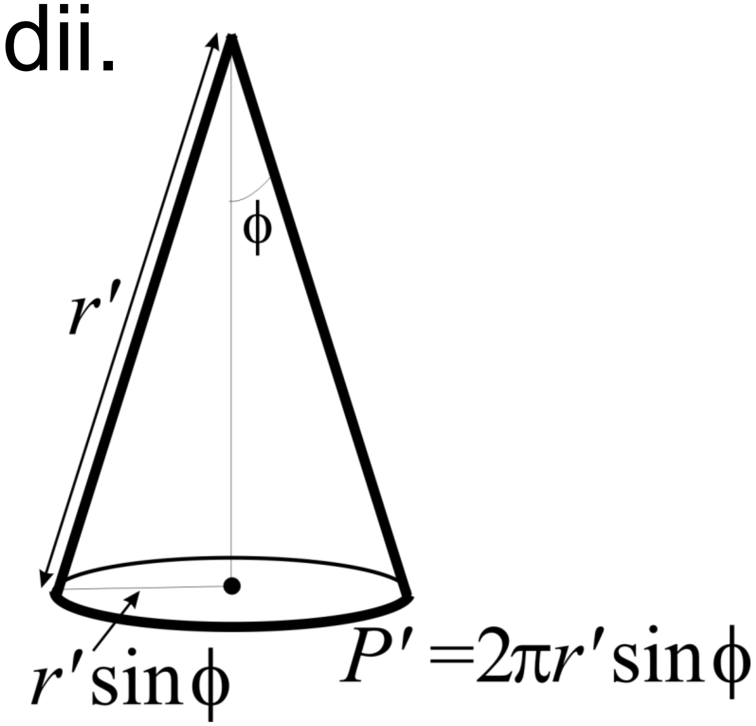


Change λ : geodesics
<u>don't</u> remain geodesics

Azimuthal director; radii remain radii.



$\lambda < 1$



$P' = \lambda 2\pi r$ contraction

$r' = \lambda^{-\nu} r$ elongation $(\rightarrow r = \lambda^{\nu} r')$

$$P' = 2\pi\lambda^{1+\nu} r'$$

but on cone

$$P' = 2\pi r' \sin\phi$$

$\boxed{\Rightarrow \sin\phi = \lambda^{1+\nu}}$ Cone angle depends on thermal contraction.

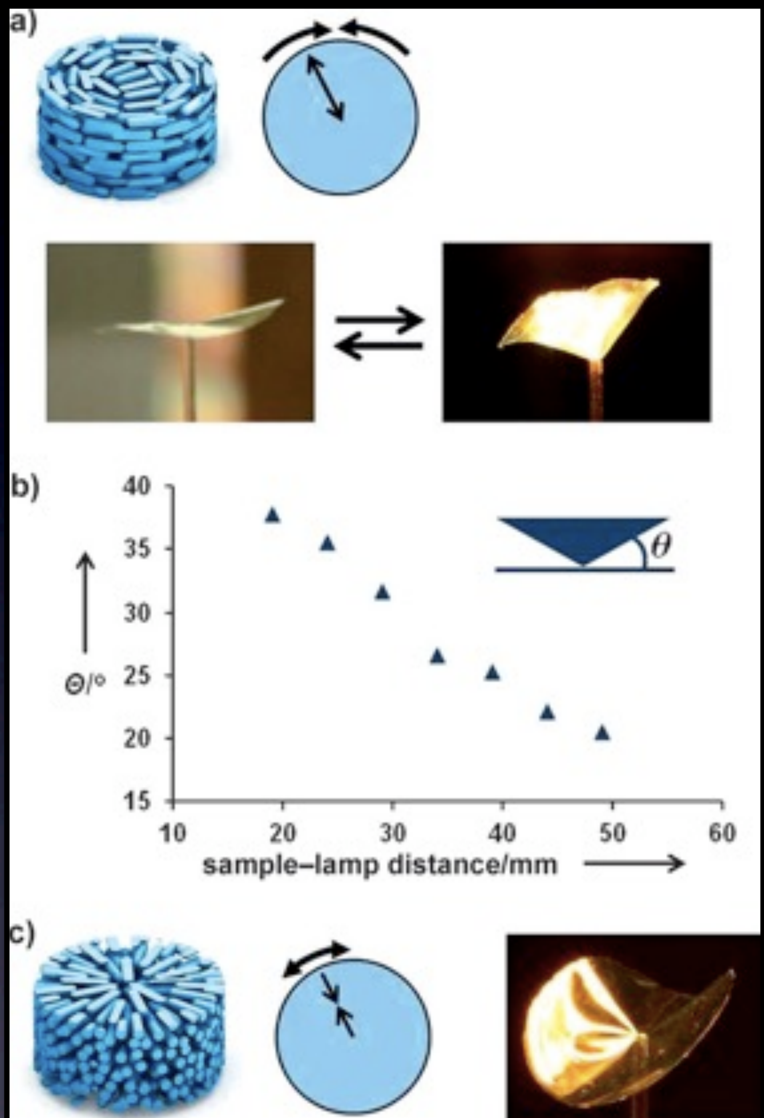
Nematic glass

$\lambda \approx .97$

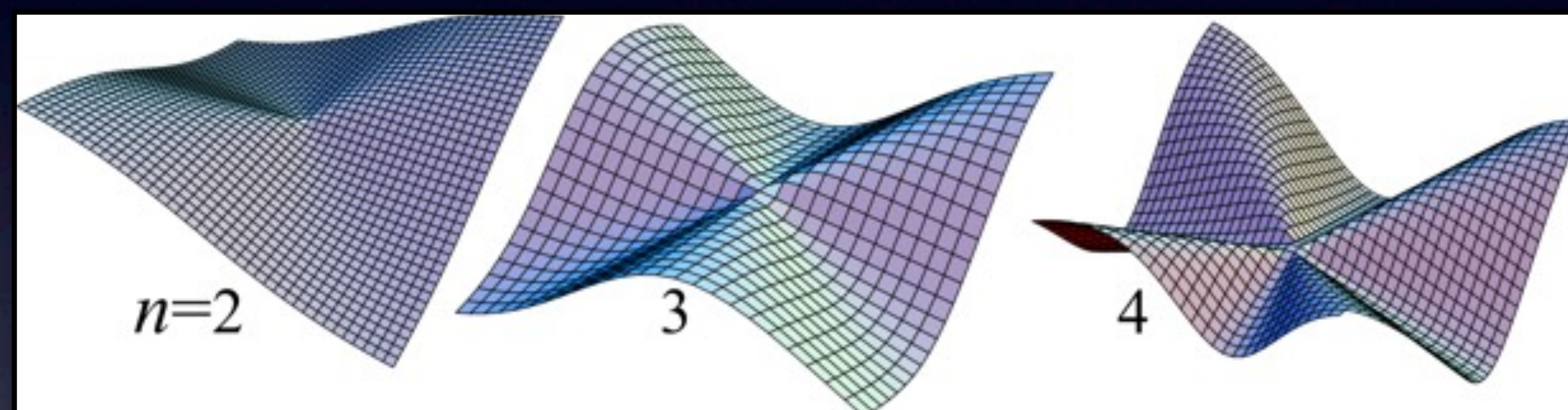
$\nu \approx 2.0$

$\phi \sim 70^{\mathrm{o}}$

Quick Refresher: Anti-Cones



L.T. de Haan, C. Sanchez-Somolinos, C.M.W. Bastiaansen, A.P.H.J. Schenning, & D.J. Broer, *Angew. Chem. Int. Ed.* **51** 12469 (2012)



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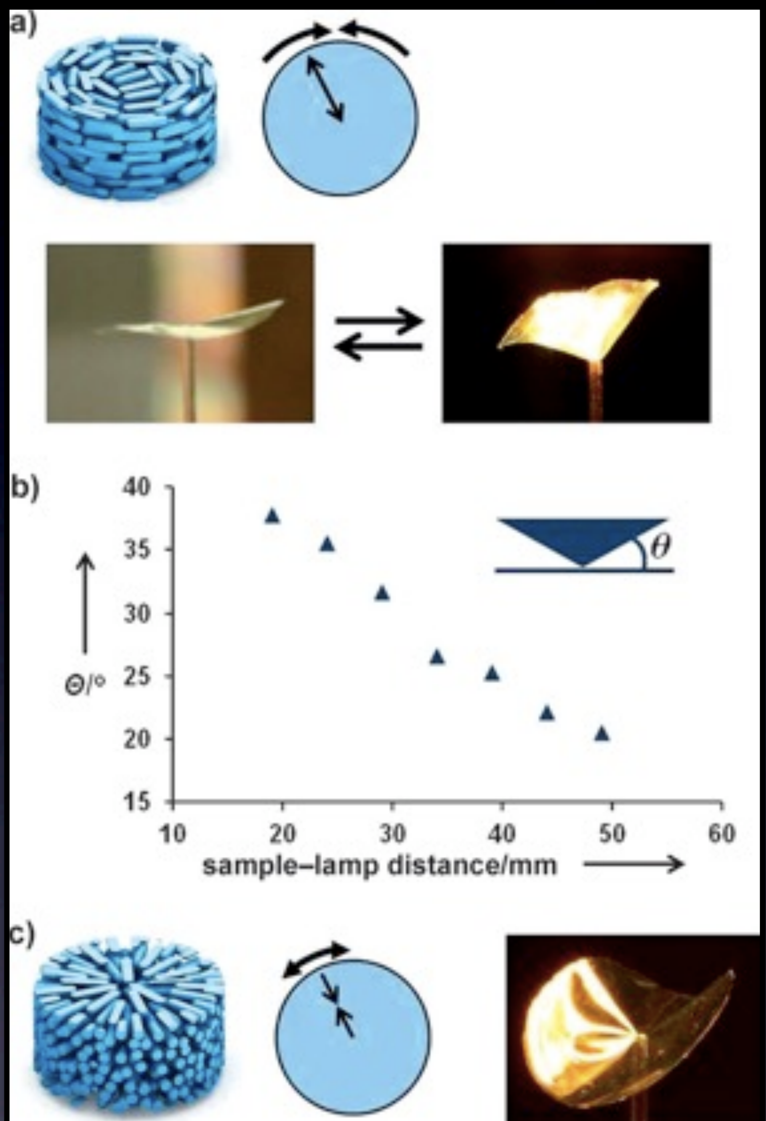

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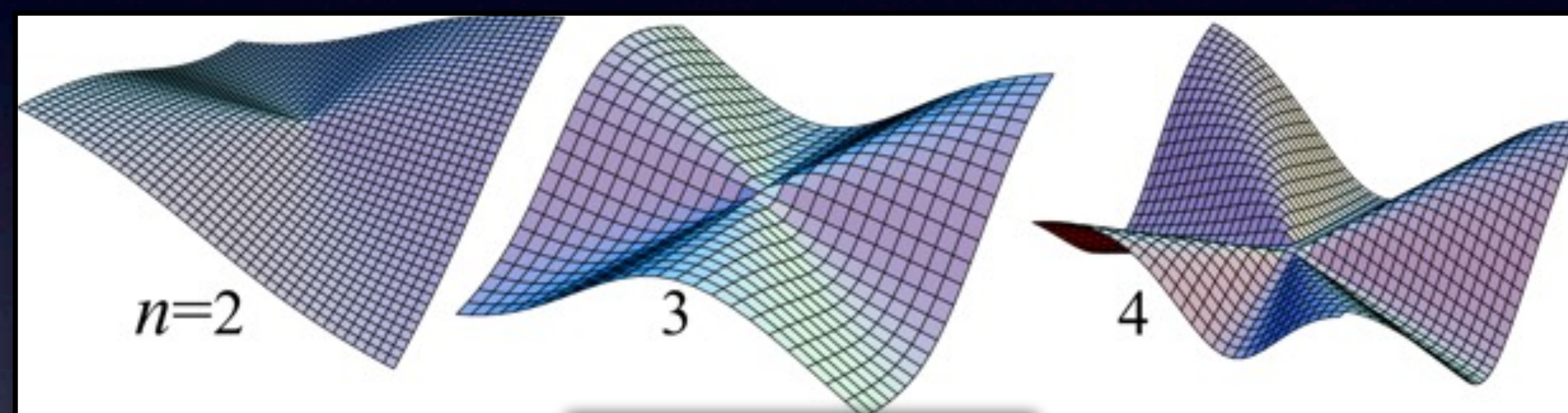
$$h(r; A, n) = Ar \sin n\phi$$



$$P' = 2\pi\lambda^{1+\nu} r'$$

Transforming to a Mathematically Analogous Problem

Darboux Frame on the surface

$$
\begin{aligned}
\boldsymbol{T}(s) &= \underline{\gamma}'(s)\,; \quad |\underline{\gamma}'| = 1 && \text{tangent} \\
\boldsymbol{u}(s) &\equiv \boldsymbol{u}(\underline{\gamma}(s)) && \text{unit surface normal} \\
\boldsymbol{t}(s) &= \boldsymbol{u}(s) \wedge \boldsymbol{T}(s) && \text{tangent normal}
\end{aligned}
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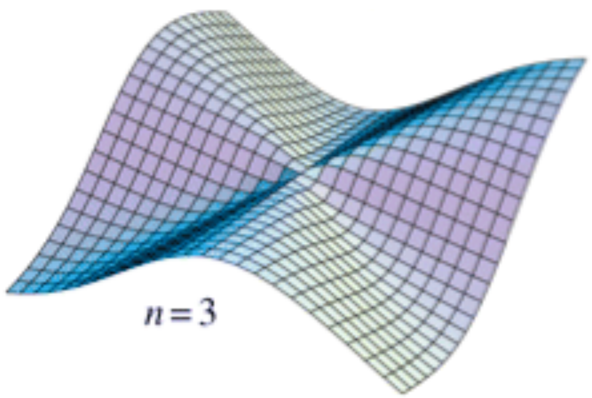

$$\boldsymbol{T}'(s) = \kappa_{\mathrm{n}} \boldsymbol{u}(s) + \kappa_{\mathrm{g}} \boldsymbol{t}(s)$$

$$\int \mathrm{d}r.r\mathrm{d}s \left((\hat{\boldsymbol{r}} \wedge \underline{\gamma}') \cdot \frac{\underline{\gamma}''}{r} \right)^2 = \int \frac{\mathrm{d}r}{r} \mathrm{d}s \left((\hat{\boldsymbol{r}} \wedge \underline{\gamma}') \cdot \underline{\gamma}'' \right)^2$$

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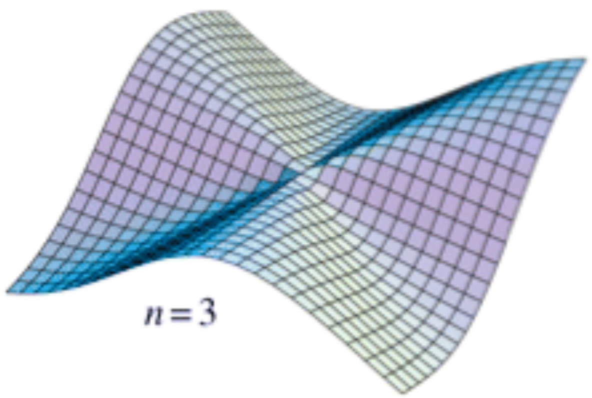

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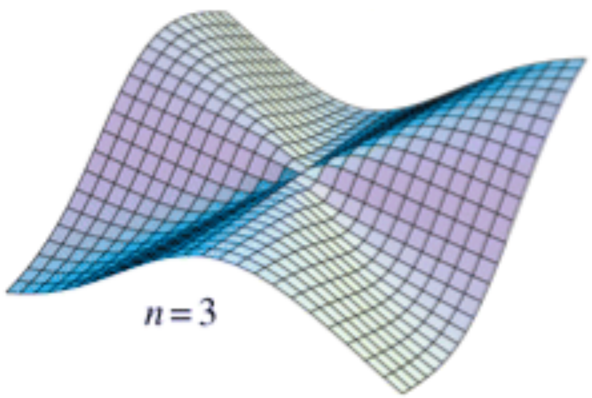

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But!

The curve as defined also lies on the surface of a sphere... what about the Darboux Frame of the curve w.r.t. the sphere?

$$\boldsymbol{u}, \boldsymbol{t} \to \boldsymbol{t}_{\mathrm{s}}, \boldsymbol{u}_{\mathrm{s}}$$

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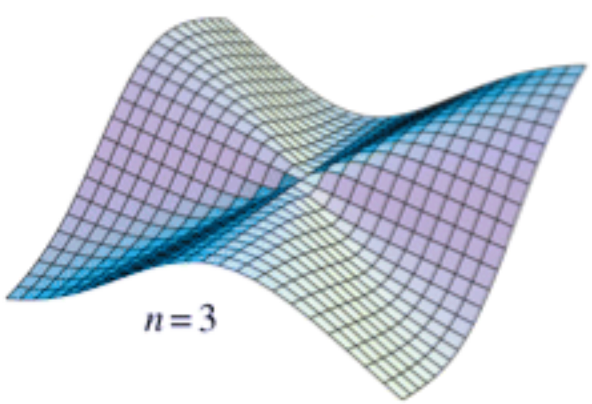

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→ New Minimization Problem

$$\int \mathrm{d}A\kappa_{\mathrm{g}}^{\mathrm{s}2}$$

This is precisely the classical problem of an elastica on the surface of a sphere...

C.D. Modes and M. Warner, *Phys. Rev. E* 92(1) 010401 (2015)