Inferring Uncertainties in Computational Anatomy

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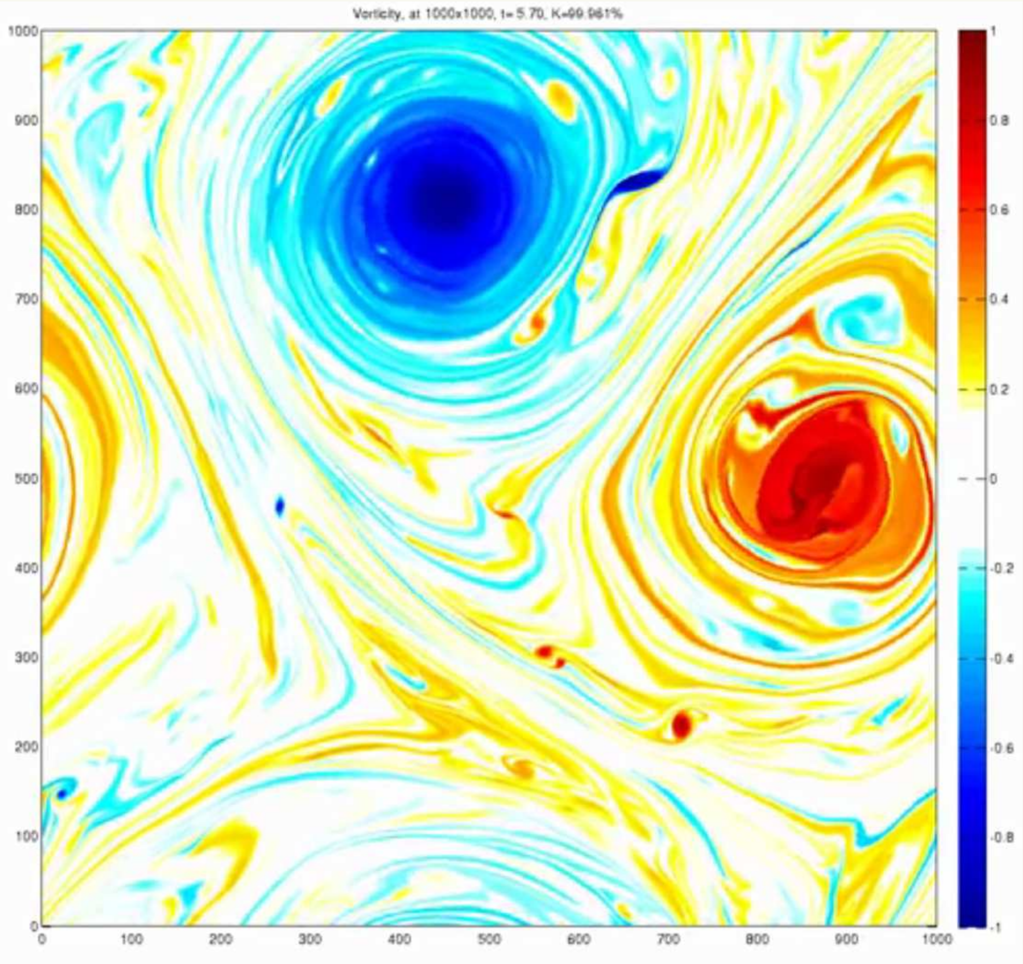
Part I: Computational anatomy as fluid dynamics

Ideal fluid (or not)

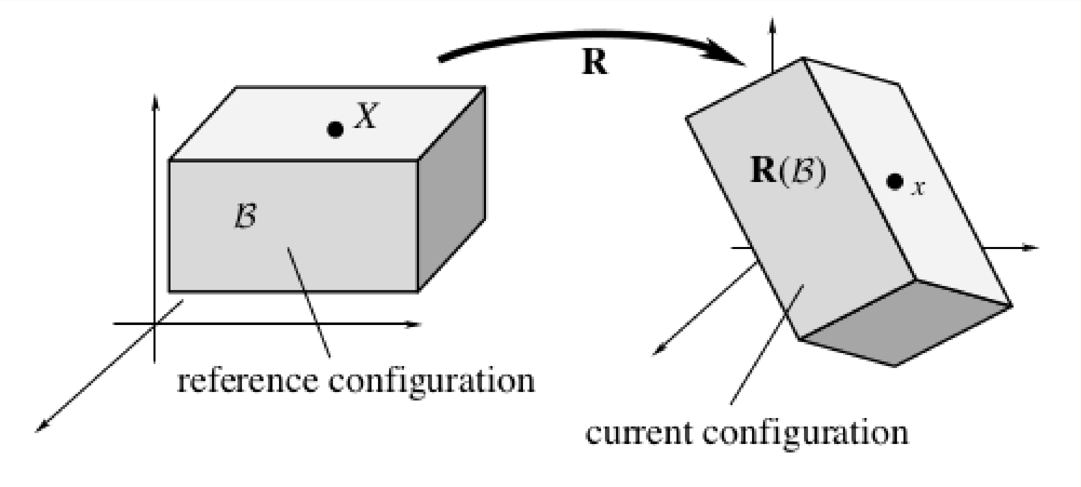


2D incompressible Euler equation for a velocity field $\mathbf{v}(x,t) \in \mathbb{R}^2$:

$$\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} = \nabla p, \quad \operatorname{div}(\mathbf{v}) = 0\,.$$

Incompressible idea fluid

Fluid is described by a flow map φ from reference configuration to the current configuration:



This theory states that $\mathbf{v}$ satisfies the Euler-Poincaré equation

$$\frac{d}{dt}\frac{\delta l(\mathbf{v})}{\partial \mathbf{v}} + \mathrm{ad}^*_{\mathbf{v}}\frac{\delta l(\mathbf{v})}{\partial \mathbf{v}} = \dot{\mathbf{v}} + \mathbf{v}\cdot\nabla\mathbf{v} = 0\,,$$

and $x = \varphi(X,t)$ is found by solving

$$\mathbf{v}(x,t) = \dot{\varphi}\circ\varphi^{-1}(x,t)\,.$$

Singular solutions (landmarks)

After averaging over small scales, we can express $\mathbf{v} = K * \mathbf{m}$:

$$K(x) \propto e^{-\frac{|x|}{2\alpha}} \quad \Rightarrow \quad \dot{\mathbf{m}} + \mathbf{v} \cdot \nabla \mathbf{m} + \mathbf{m}_j \nabla \mathbf{v}^j = \nabla \pi \,.$$

Pick a set of points $\mathbf{q}_i$ with momentum $\mathbf{p}_i$

$$\mathbf{m}(x,t) = \sum_i \mathbf{p}_i(t) \delta(x - \mathbf{q}_i(t)), \,.$$

We obtain the set **exact** set of Hamiltonian dynamical equations

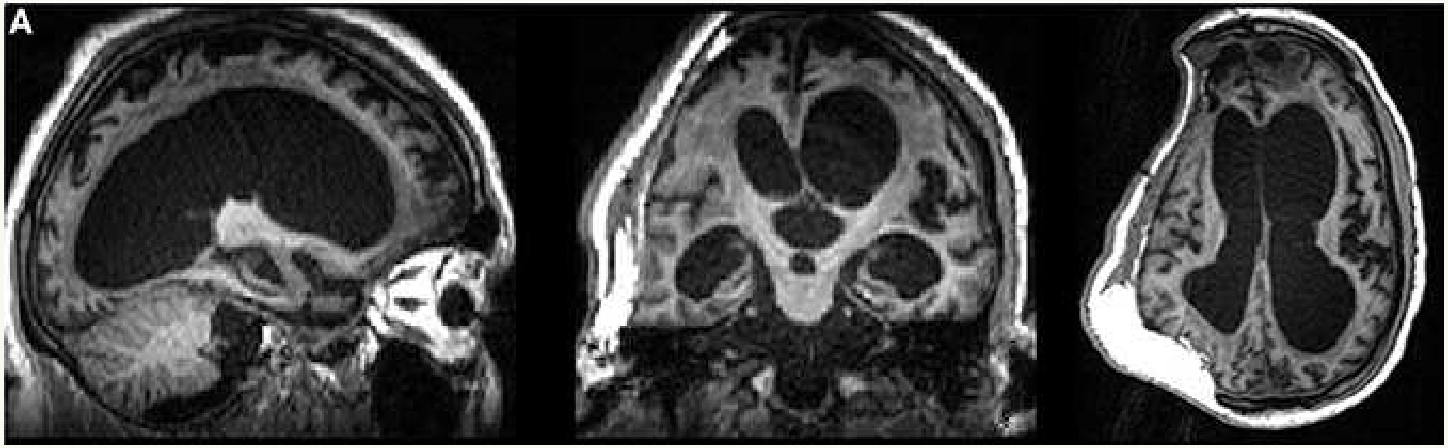
$$\dot{\mathbf{q}}_i = \frac{\partial h}{\partial \mathbf{p}_i} = \sum_j \mathbf{p}_j K(\mathbf{q}_i - \mathbf{q}_j)$$

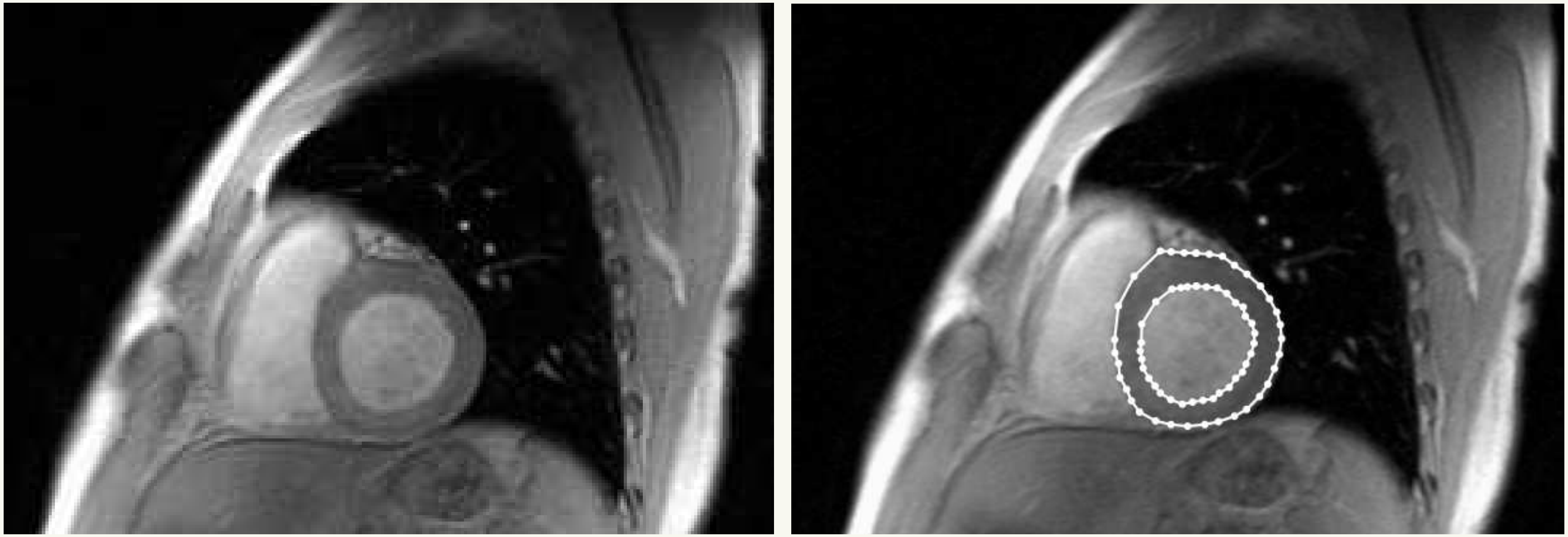
$$\dot{\mathbf{p}}_i = -\frac{\partial h}{\partial \mathbf{q}_i} = -\sum_j \mathbf{p}_i \cdot \mathbf{p}_j \frac{\partial}{\partial \mathbf{q}_i} K(\mathbf{q}_i - \mathbf{q}_j) \,,$$

$$\text{where} \quad h(\mathbf{q}, \mathbf{p}) = \frac{1}{2} \sum_{i,j=1}^{N} (\mathbf{p}_i \cdot \mathbf{p}_j) K(\mathbf{q}_i - \mathbf{q}_j) \,.$$

Part II: Shape matching

Large deformation matching



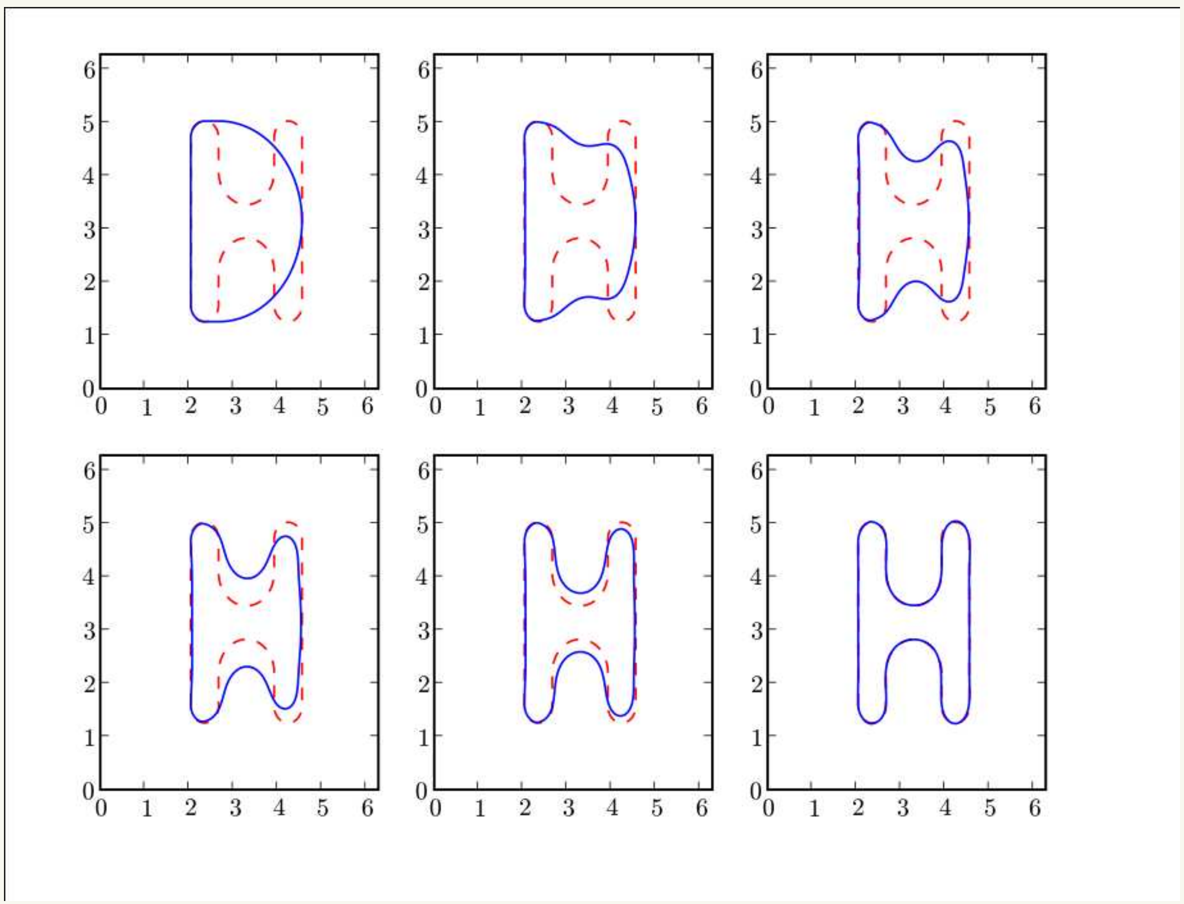


Optimisation problem

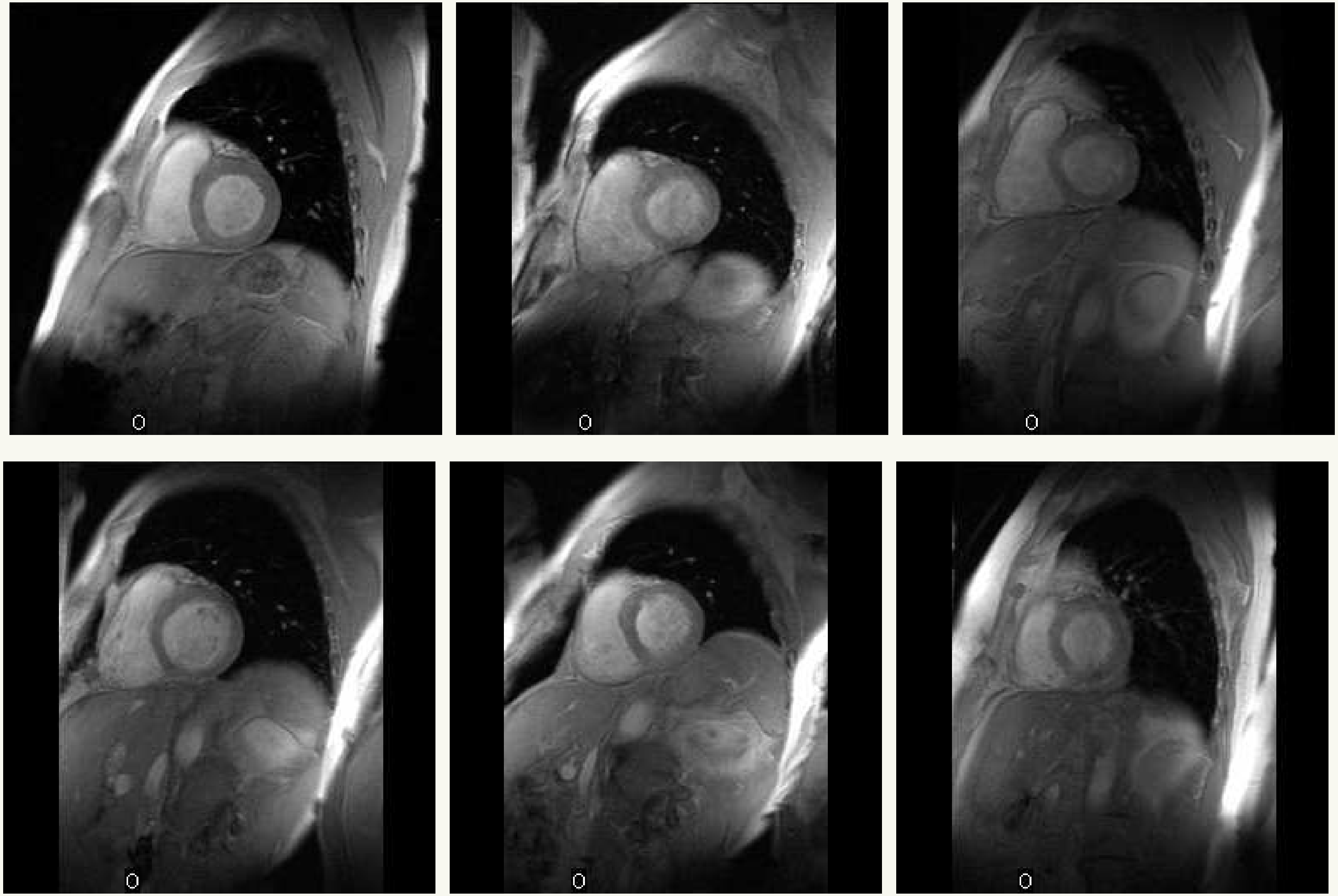
Given $\mathbf{q}_0$ and $\mathbf{q}_T$, find $\mathbf{p}(0)$ satisfying

$$\min_{\mathbf{p}(0)} \int_0^1 l(\mathbf{v})\,dt$$

$$\text{such that} \quad \mathbf{q}(0) = \mathbf{q}_0 \quad \text{and} \quad \mathbf{q}(1) = \mathbf{q}_T\,.$$



Variability in images



Part III: Modelling uncertainties

Geometric noise vs. additive noise

1. Stochastic model of this work:

$$d\mathbf{q}_i = \frac{\partial h}{\partial \mathbf{p}_i} dt + \sum_l \sigma_l(\mathbf{q}_i) \circ dW_t^l \,,$$

$$d\mathbf{p}_i = -\frac{\partial h}{\partial \mathbf{q}_i} dt - \sum_l \frac{\partial}{\partial \mathbf{q}_i} \left(\mathbf{p}_i \cdot \sigma_l(\mathbf{q}_i)\right) \circ dW_t^l \,.$$

$\Rightarrow$ **The noise is 'Eulerian', attached to the image.**

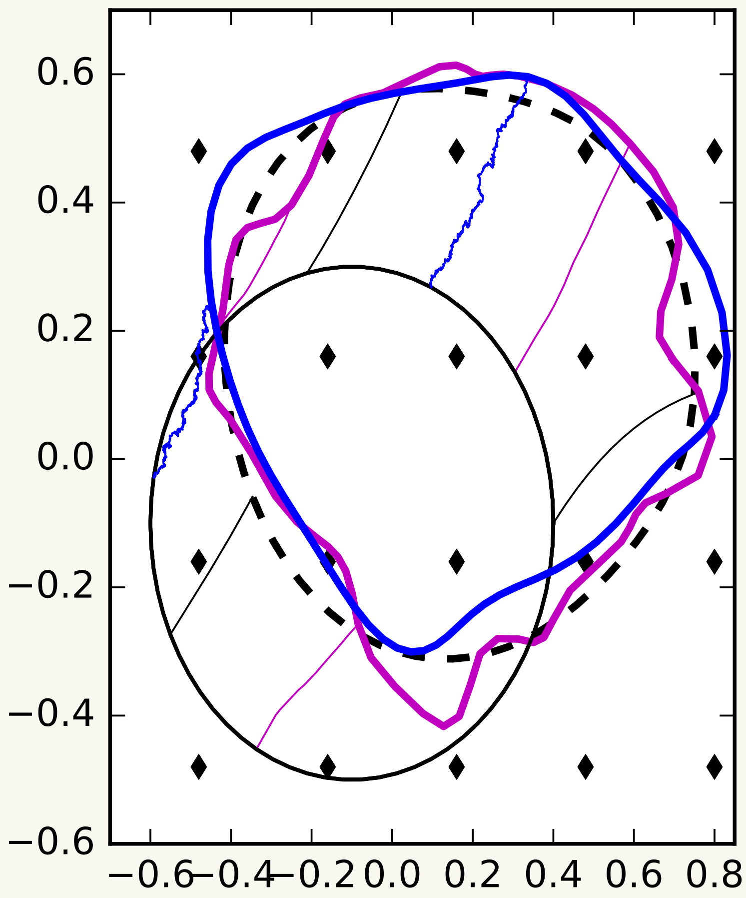
2. Stochastic forcing model ($\sigma_i \in \mathbb{R}$):

$$d\mathbf{q}_i^\alpha = \frac{\partial h}{\partial \mathbf{p}_i^\alpha} dt$$

$$d\mathbf{p}_i^\alpha = -\frac{\partial h}{\partial \mathbf{q}_i^\alpha} dt + \sigma_i dW_t^i \,.$$

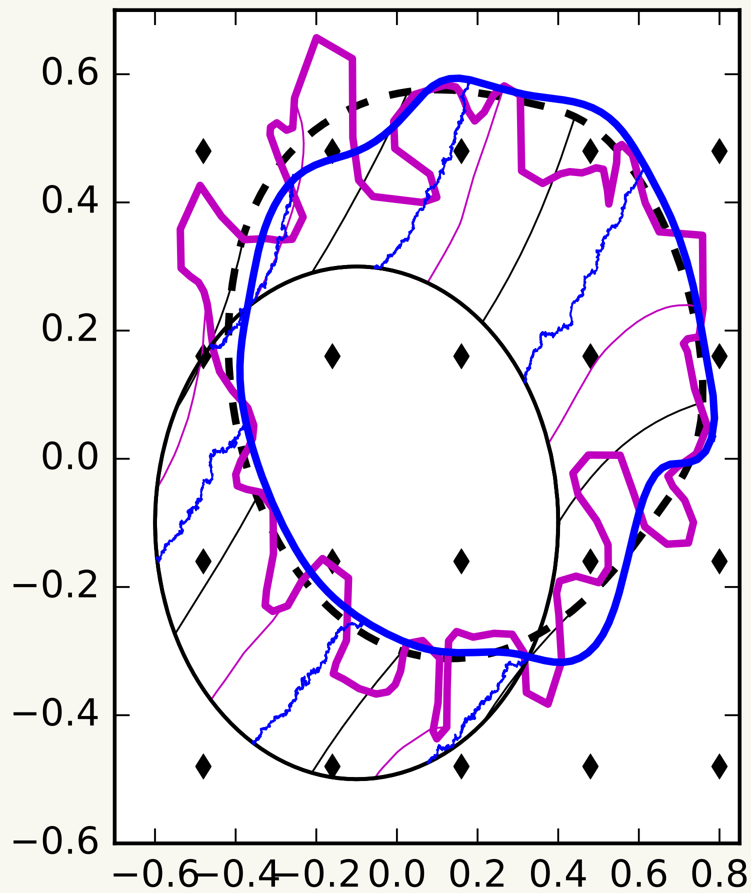
$\Rightarrow$ **The noise is 'Lagrangian', attached to the landmarks.**

Similar solutions



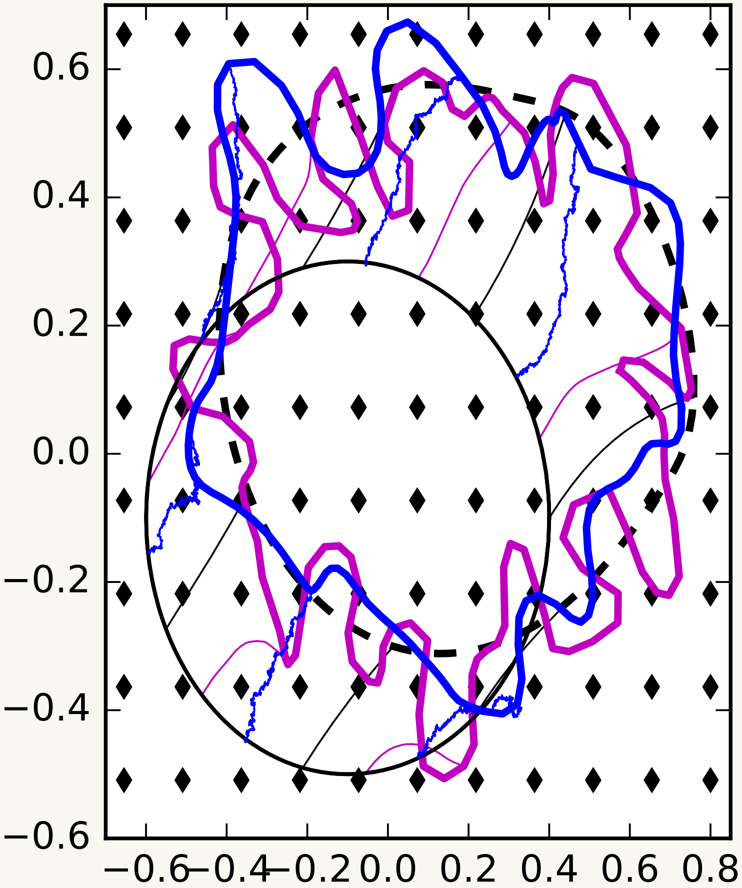
Low resolution and large noise correlation with 100 landmarks and 6×6 noise fields.

Double spacial resolution



High resolution and large noise correlation with 200 landmarks and 6×6 noise fields.

Half of noise correlation



High resolution and small noise correlation with 200 landmarks and 12×12 noise fields.

Part IV: Inferring the noise

Fokker-Planck equation and moments

We seek the time evolution of the probability distribution of landmarks. It is given by the Fokker-Planck equation:

$$\frac{d}{dt}\mathbb{P}(\mathbf{q},\mathbf{p},t) = \{h,\mathbb{P}\}_{\mathrm{can}} + \frac{1}{2}\sum_l \{\phi_l, \{\phi_l, \mathbb{P}\}_{\mathrm{can}}\}_{\mathrm{can}} =: \mathcal{L}^*\mathbb{P},$$

where

$$\{F,G\}_{\mathrm{can}} = \begin{pmatrix}\partial_{q_i}F & \partial_{p_i}F\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\partial_{q_i}G\\ \partial_{p_i}G\end{pmatrix}$$

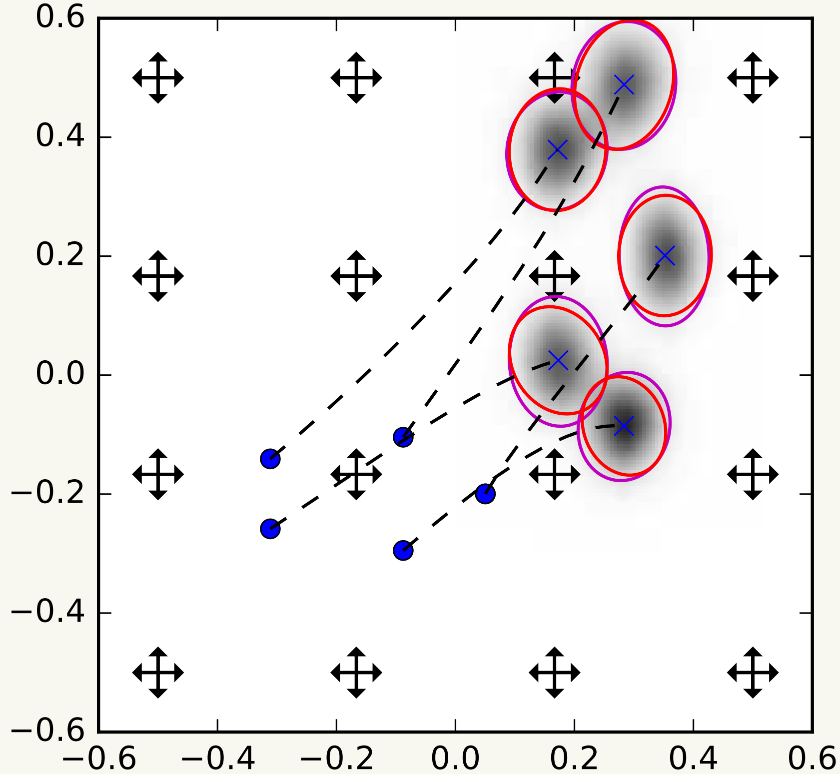
is the canonical bracket and $\phi_l(\mathbf{q},\mathbf{p}) = \sum_i \mathbf{p}_i \cdot \sigma_l(\mathbf{q}_i)$. This is a very high dimensional PDE, and we are only interested in these two quantities:

$$\langle q_i^\alpha\rangle(t) := \int q_i^\alpha \mathbb{P}(\mathbf{q},\mathbf{p},t)\, d\mathbf{q}d\mathbf{p}$$

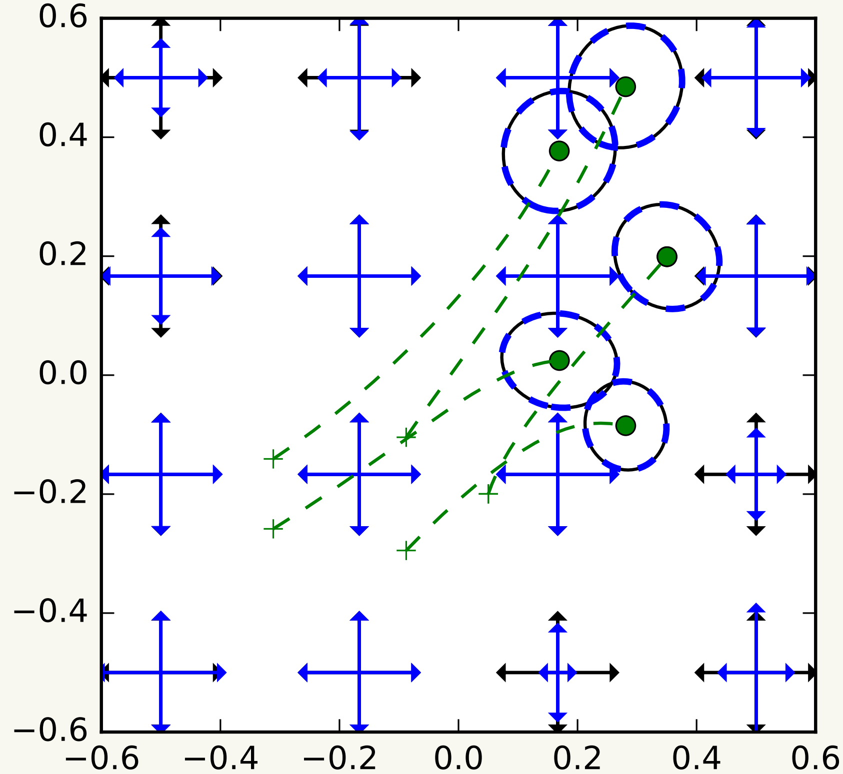
$$\langle q_i^\alpha q_j^\beta\rangle(t) := \int q_i^\alpha q_j^\beta \mathbb{P}(\mathbf{q},\mathbf{p},t)\, d\mathbf{q}d\mathbf{p},$$

called the moments of the Fokker-Planck equation.

Moment evolution

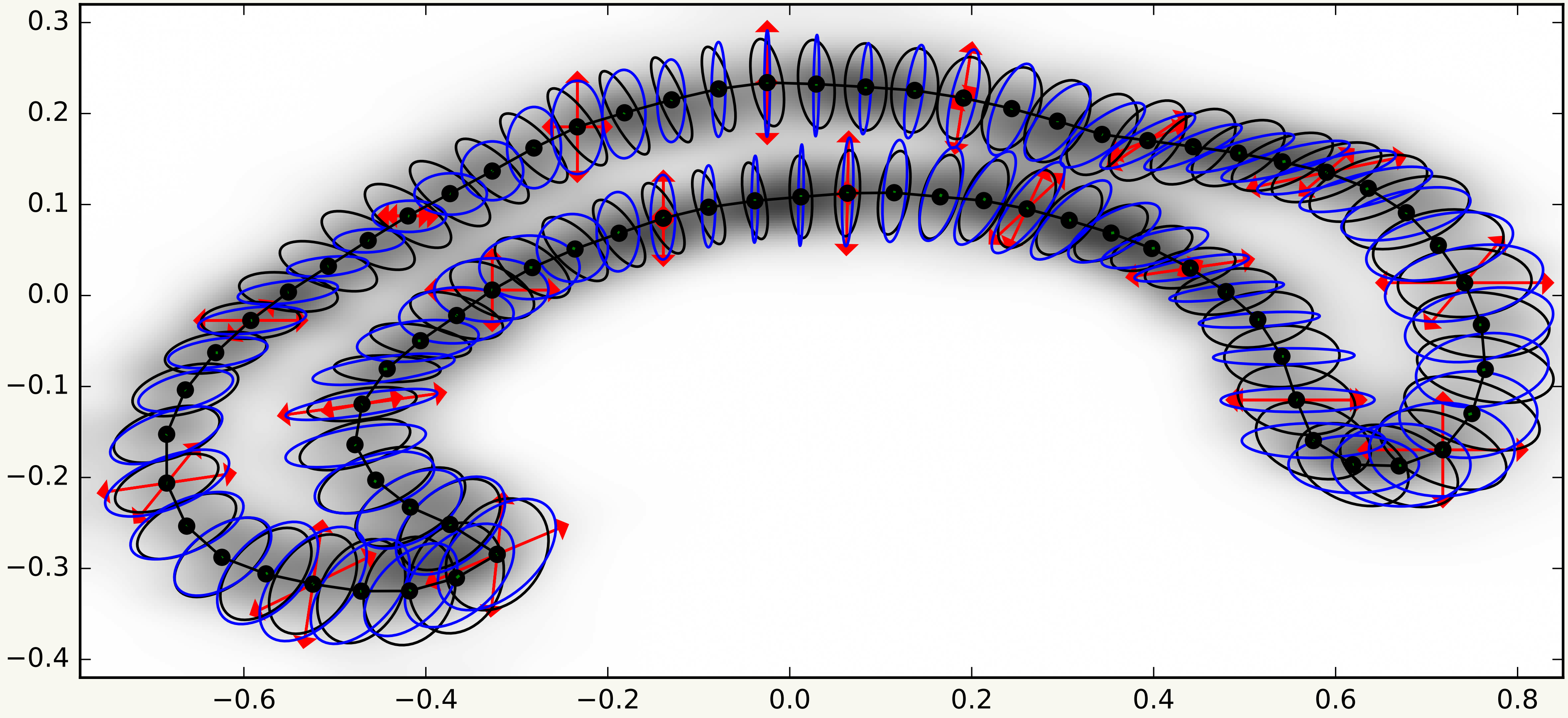


(i) Monte Carlo vs. Moment equations



(j) Inverse problem with a genetic algorithm

Application for Corpus Callosum



From 50 Corpus Callosum shapes with variance in black, the model with 18 noise fields in red reproduces the variability in blue.

Main references

Arnaudon, A., Holm, D. D., Pai, A., and Sommer, S. A stochastic large deformation model for computational anatomy. In *Information Processing for Medical Imaging (IPMI)* (2017).

Arnaudon, A., Holm, D. D., and Sommer, S. A Geometric Framework for Stochastic Shape Analysis. *arXiv:1703.09971, to appear in Foundation of Computational Mathematics* (2018).

Holm, D. D. Variational principles for stochastic fluid dynamics. *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences 471*, 2176 (2015), 20140963.