ODE and PDE Based Modelling of Biological Transportation Networks

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Discrete model (Hu, Cai)

- Flow of a material through the **network graph** $(\mathcal{V}, \mathcal{E})$:

  Pressures P_j on vertices $j \in \mathcal{V}$

  Conductivities C_{jk} on edges $(j, k) \in \mathcal{E}$

- Fluxes:

$$Q_{jk} = C_{jk} \frac{(\Delta P)_{jk}}{L_{jk}}$$

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- **Kirchhoff law (conservation of mass)** with sources S_j:

$$\sum_{k \in N(j)} Q_{jk} = \sum_{k \in N(j)} (P_j - P_k) \frac{C_{jk}}{L_{jk}} = S_j \qquad \text{for all } j \in \mathcal{V}$$

  for set $N(j)$ of vertices adjacent to vertex j

- **Energy cost functional**:

$$E_{\text{disc}}[C] = \sum_{(j,k) \in \mathcal{E}} \left(C_{jk} \left(\frac{(\Delta P_{jk})}{L_{jk}} \right)^2 + \frac{\nu}{\gamma} C_{jk}^{\gamma} \right) L_{jk}$$

Connection between the discrete and the continuum model

Construction of continuum energy minimizers

- **Regularisation and reformulation** of the discrete model so that energy functional in integral form with added diffusion
- **Continuum model:**

$$\mathbb{E}[c] = \int_\Omega D^2 |\nabla c|^2 + \nabla p \cdot (r\mathbb{I} + c)\nabla p + \frac{\nu}{\gamma}|c|^\gamma \,\mathrm{d}x,$$

where $p[c]$ is a weak solution of the Poisson equation

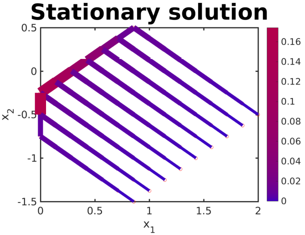
$$-\nabla \cdot ((r\mathbb{I} + c)\nabla p) = S$$

- **Convergence proof**

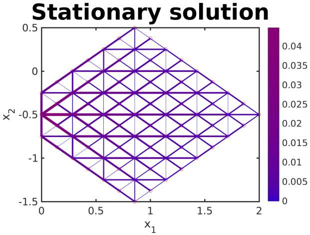
Steady states

Dependence on γ

- $\gamma \in \mathbf{(0,1)}$: All local minima of the discrete model are trees, i.e. trees are stable steady states.
- $\gamma > \mathbf{1}$: The graph associated with any local minimum contains loops. In particular, a tree cannot be steady state.

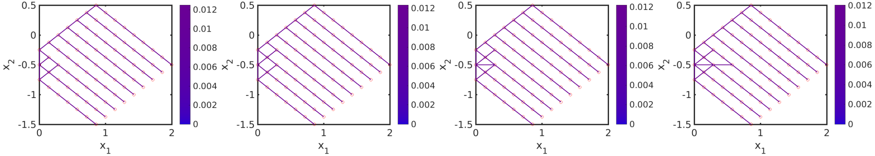


(a) $\gamma = 0.5$

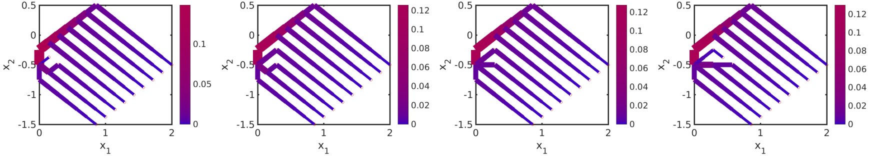


(b) $\gamma = 1.5$

Stability of steady states when several loops in tree-structured initial data are closed in the discrete model



(a) Initial data



(b) Associated steady states

Conclusion

Results and future work

- **Existence of solutions** to the discrete and continuum model
- Rigorous proof of the **continuum limit** of the discrete model via Γ-convergence
- **Numerical analysis** of steady states and their stability
- **Modelling and simulation of leaf venation networks** in collaboration with The Sainsbury Lab, University of Cambridge

References

- J. Haskovec, LMK and P. Markowich, *ODE and PDE based modeling of biological transportation networks*, arXiv:1805.08526, submitted to CMS, 2018.
- J. Haskovec, LMK and P. Markowich, *Continuum Limit for the Discrete Network Formation Problem*, in preparation.

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Thank you for your attention!