Analysis of nonlinear equations in the power grid

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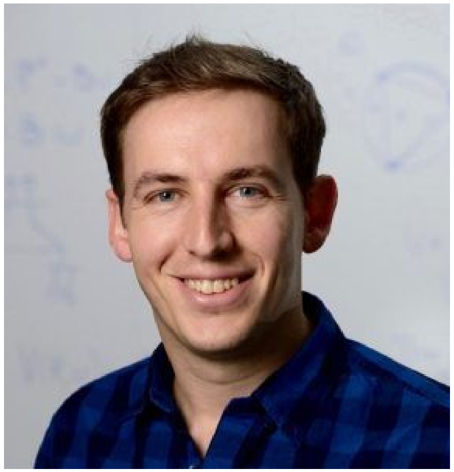
Acknowledgements



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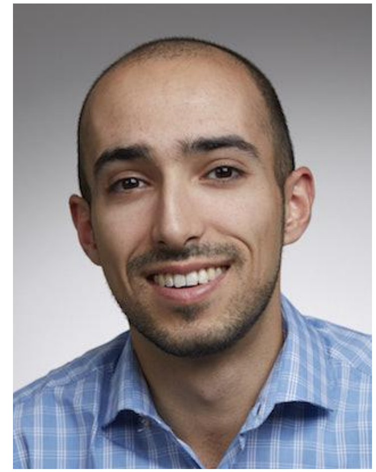

John Simpson-Porco

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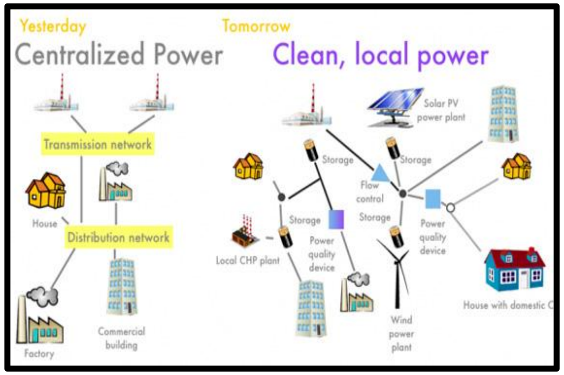
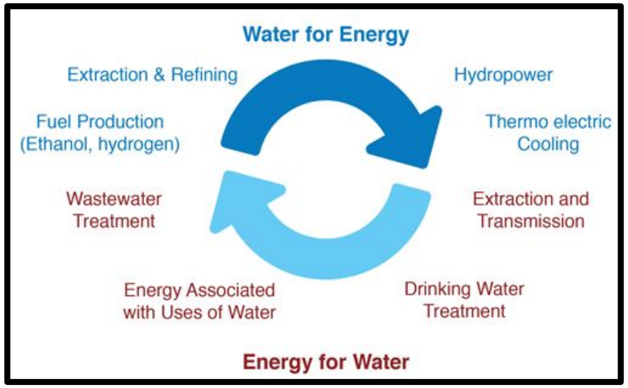
Emerging challenges in the power grid

Weather-dependent
Renewable generation



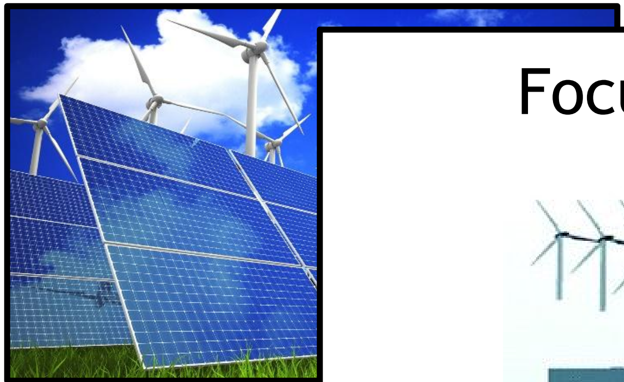
Distributed & decentralized system



Interdependence: Water, Gas, Power Transportation

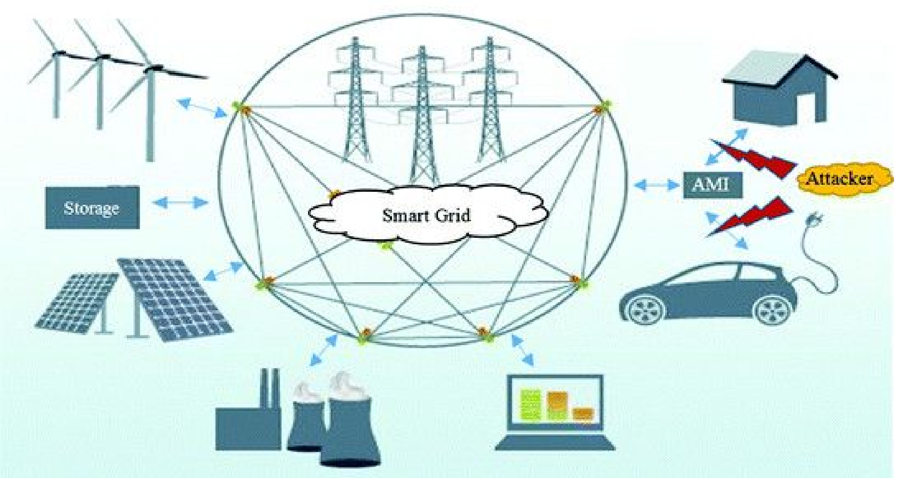
<u>Challenges</u>: Increased uncertainty, complex coupling, decentralization of resources

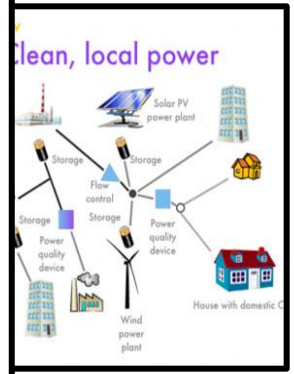
Emerging challenges in the power grid



Weather-d
Renewable

Focus on distribution systems





uted &
tralized
tem

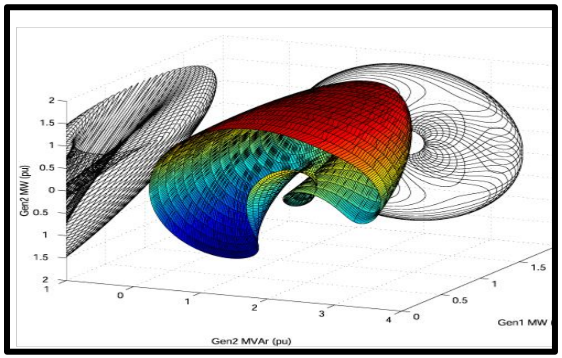
- Rapid increase in “active resources”
- Increased monitoring capabilities (opportunity)
- New flow patterns (reverse flows, stochastic, distributed. gen., etc.)

need to make optimal use of resources requires better tools to characterize the system security

<u>Challenges</u>: Increased uncertainty, complex coupling, decentralization of resources

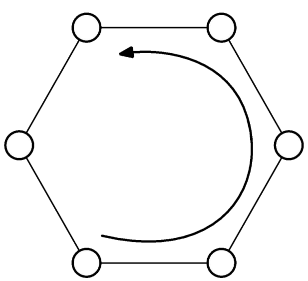
Power flow equations

Relate power injections with steady-state voltages magnitudes and angles



Nonlinearity and
Non-convexity

Power flow solutions can behave strangely

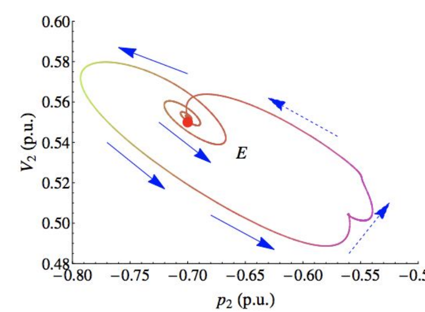


Loop flows
[Bergen Vittal , Mallada '11]

Analysis of power-flow equation

A Araposthatis, S Sastry and P Varaiya
Department of Electrical Engineering and Computer Sciences and Electronics Research Laboratory, University of California, Berkeley, CA 94720, USA

Can have only
unstable solutions!
[Sastry, Varaiya '81]

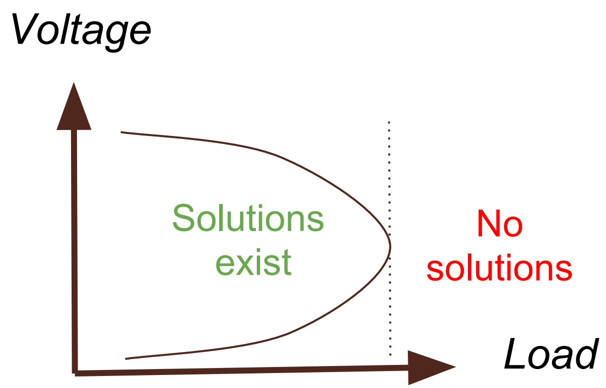


Multi-stability
[Turitsyn '16]

yet, the grid tend to operate on a "high-voltage" solution

Power flow equations

Lack of solution ⇨ **Voltage Collapse**



Finding solutions is critical for reliable operation!

- security assessment
- optimal power flow
- state estimation
- stability analysis
- controller design
- much more…

Solving power flow equations

Traditional algorithms (Newton-Raphson, Gauss-Seidel, etc.)

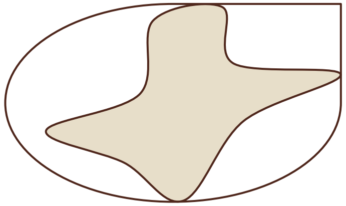
- "usually" work well…
- no clear reason for failure: initialization?, num. errors?, existence?
- introduces conservativeness

More recently…

Fixed-point iterations
[Turitsyn '15, Bolognani '16, Paolone '16, Simpson-Porco '17,…]

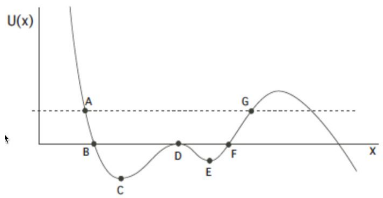
$$v^+ = g(v)$$

Convex relaxations
[Jabr '06, Lavaei '12, …]



generally provide sufficient OR necessary conditions

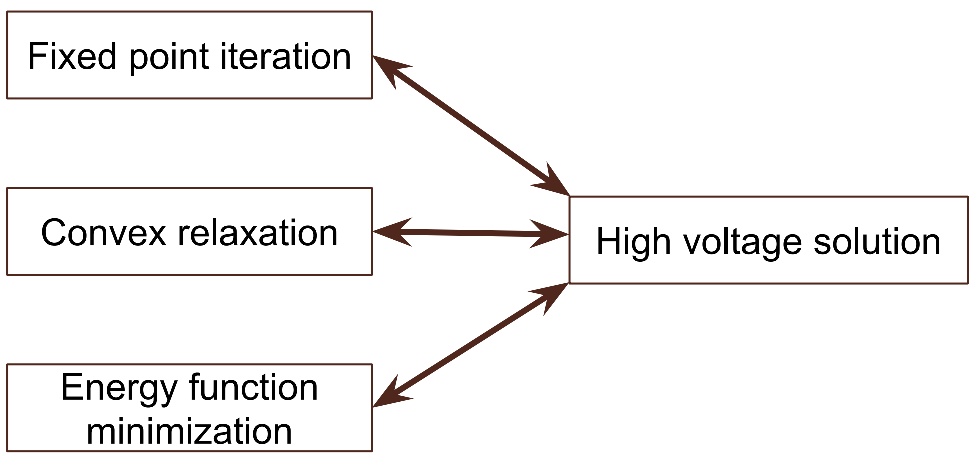
Energy function minimization
[Dvijotham, Low, Cherktov '15]



Contributions of this work

Tight necessary and sufficient conditions for existence of power flow solutions in a balanced homogeneous distribution system

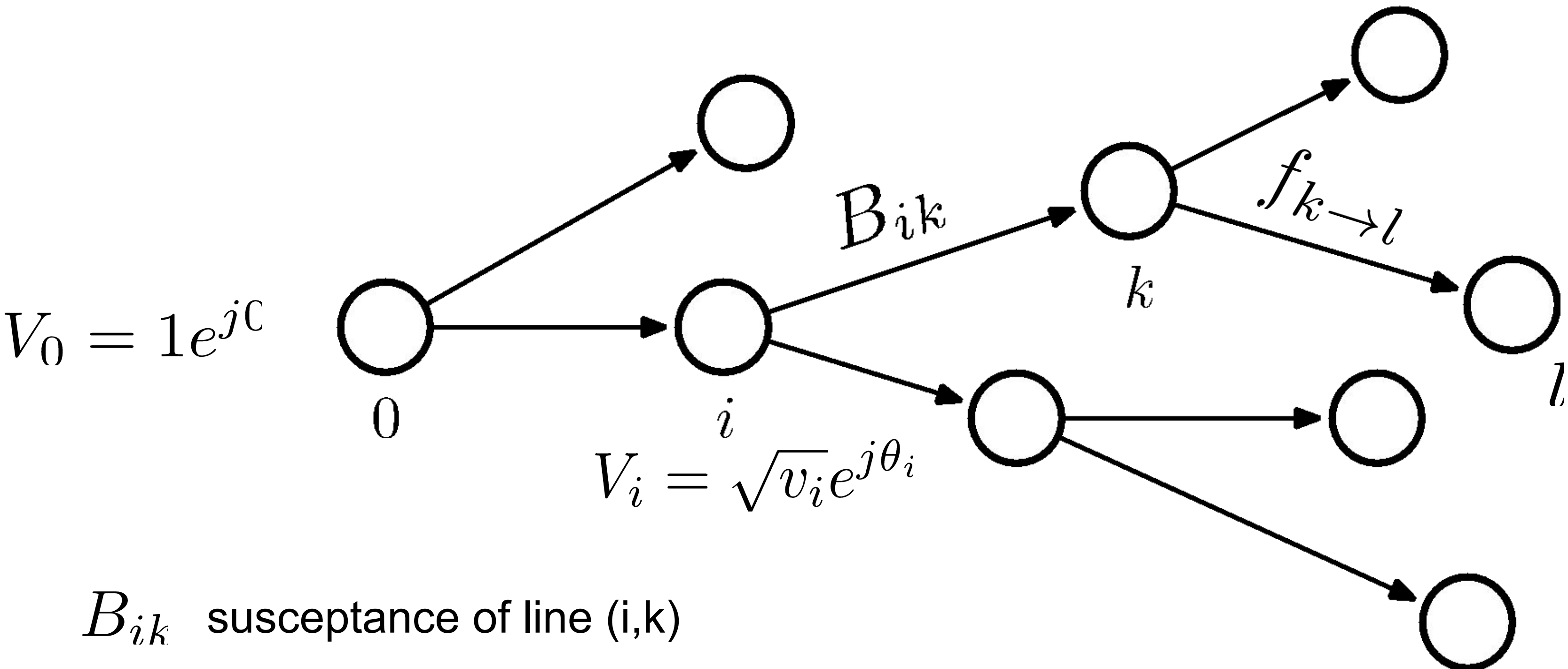
- Provide visibility on exact limits of distribution system
- Unified viewpoint on multiple power flow solution algorithms
- Establish existence, uniqueness and desirable properties of "high-voltage" power flow solution



1. PF solution exists -> Unique high voltage solution and all algorithms converge to it
2. When algorithms fail -> no solution exists

Contribution: First analytical understanding of tight necessary and sufficient conditions for PF

Power flow for tree networks



B_{ik} susceptance of line (i,k)

v_i voltage magnitude squared

θ_i voltage phase

$f_{k\to l}$ real power flow from k to l

$$f_{k\to l} = B_{kl}\sqrt{v_k v_l}\sin(\theta_k - \theta_l)$$

Assumptions:

- lossless (or constant r/x ratio)
- no shunt elements!
- balanced operation (single phase)

Power flow for lossless tree networks

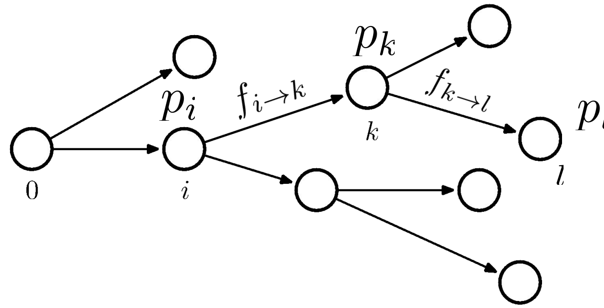
$$\sum_k f_{i\to k} = p_i,\, f_{i\to k} = -f_{k\to i}$$

Lossless active balance

$$\sum_k B_{ik}\left(v_i - \sqrt{v_i v_k - \left(\frac{f_{ik}(p)}{B_{ik}}\right)^2}\right) = q_i$$

Lossy reactive balance

Reactive flow
from i->k



Since network is tree, active balance can be solved explicitly: $f_{ik} = f_{ik}(p)$

Left with single equation per bus: $$\sum_k B_{ik}\left(v_i - \sqrt{v_i v_k - \left(\frac{f_{ik}(p)}{B_{ik}}\right)^2}\right) = q_i$$

<u>Conclusion</u>: Power flow equations for lossless tree networks simplified to single nonlinear equation per bus

Three different interpretations

$$\sum_k B_{ik}\left(v_i - \sqrt{v_i v_k - \left(\frac{f_{ik}(p)}{B_{ik}}\right)^2}\right) = q_i$$

Relax

Rewrite

First integral

$$\sum_k B_{ik}\left(v_i - \sqrt{v_i v_k - \left(\frac{f_{ik}(p)}{B_{ik}}\right)^2}\right) \leq q_i$$

Convex relaxation

$$\sum_k \frac{B_{ik}}{\sum_{k'} B_{ik'}} \sqrt{v_i v_k - \left(\frac{f_{ik}(p)}{B_{ik}}\right)^2} + \frac{q_i}{\sum_k B_{ik}} = v_i^+$$

Fixed-point form

$$\frac{\partial E}{\partial \log(v)} = 0$$

Energy function

Partial version of complete energy function of v, θ

Conclusion: Each formulation of the PF equations leads to different approach to computing PF solutions

Power flow by convex optimization

$$\max \sum_i w_i \log(v_i)$$

$$\text{Subject to } \sum_k B_{ik} \left(v_i - \sqrt{v_i v_k - \left(\frac{f_{ik}}{B_{ik}}\right)^2} \right) \leq q_i \quad \forall i \in \{1, \ldots, n\}$$

- Convex optimization problem: Second order cone program ☺ [Jabr '06]
- Objective tries to find "high voltage" solutions: Pushes relaxation to be tight ☺

<u>Conclusion</u>: Convex relaxation + voltage maximization ->
Gets power flow solutions

Power flow by fixed point iterations

$$\sum_k \frac{B_{ik}}{\sum_{k'} B_{ik'}} \sqrt{v_i v_k - \left(\frac{f_{ik}}{B_{ik}}\right)^2} + \frac{q_i}{\sum_k B_{ik}} = g_i(v) = v_i \quad \forall i \in \{1, \ldots, n\}$$

$$\boxed{v(t+1) = g(v(t)), \quad v(0) = v^{\max}}$$

$g : \mathrm{D} \mapsto \mathrm{R}^n$ is monotone in v

$$v - v' \geq 0 \implies g(v) - g(v') \geq 0$$

Knaster-Tarski Fixed Point Theorem

- If power flow solution exists: then g has “maximal” and “minimal” fixed point
- Highest voltage (least losses) and lower voltage (maximum losses) solutions exist

Conclusion: Fixed point iterations converge to unique “maximal” PF solution

Power flow by energy minimization

$$E(v,\theta) = \sum_{i,k>i} B_{ik}(v_i + v_k - 2\sqrt{v_i v_k}\cos(\theta_i - \theta_k) - \sum_i \left(p_i\theta_i + q_i\frac{\log(v_i)}{2}\right)$$

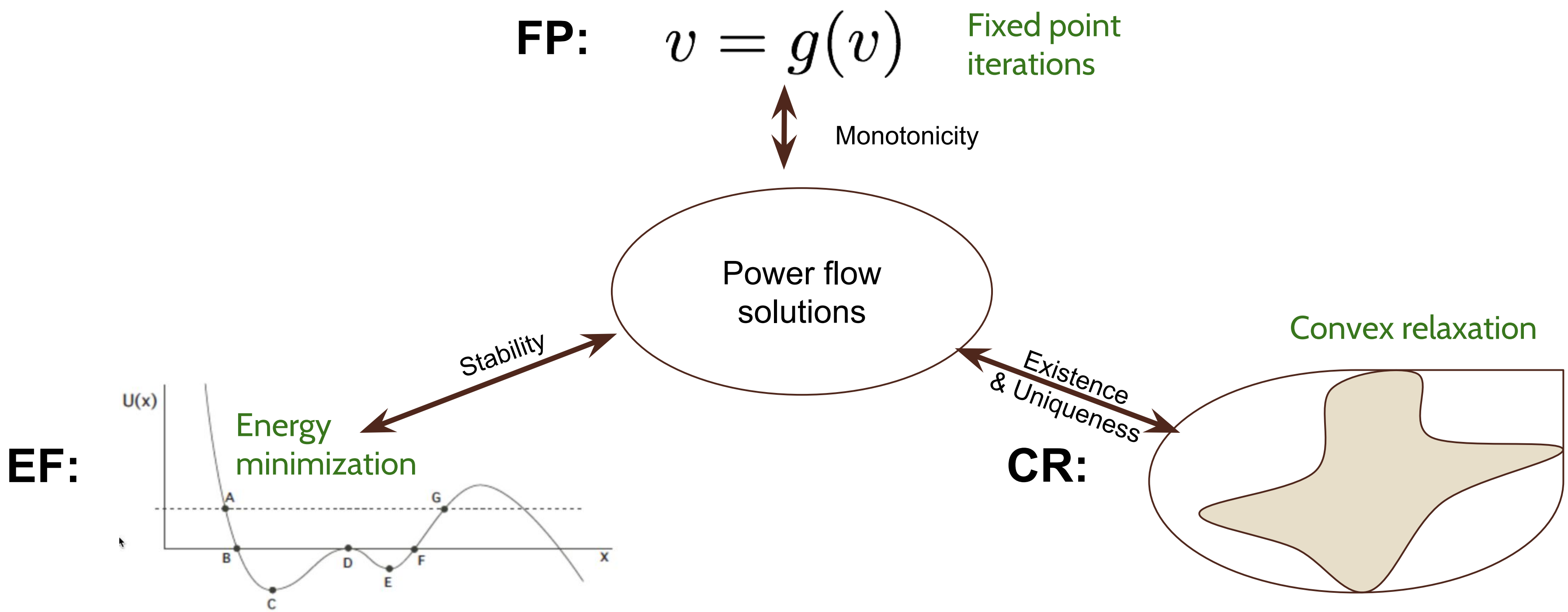
Power flows are given by:

$$\frac{\partial}{\partial \log(v)}E \equiv 0, \qquad \frac{\partial}{\partial\theta}E \equiv 0$$

- E not globally convex
- Convex over limited domain: Set of “stable” power flow solutions
- Can be found via convex optimization again ☺

<u>Conclusion</u>: Energy minimization establishes stability of PF solution

Connection between methods

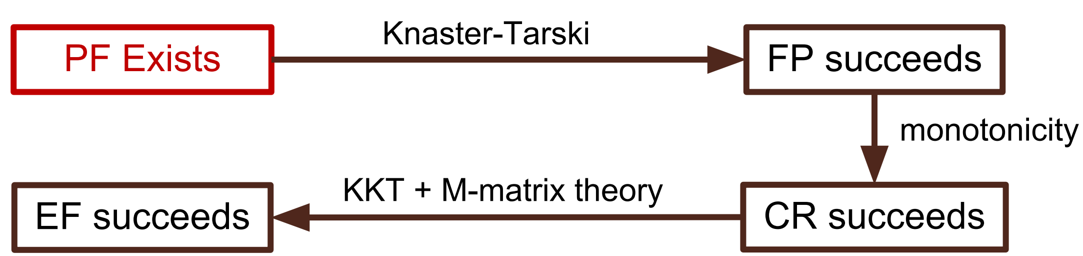


Theorem:

1. **(A)** Exists a power flow solution $\iff$ **(B)** FP, EF and CR succeed

Proof:

(A) $\implies$ **(B)**



Connection between methods

FP: $v = g(v)$ Fixed point iterations

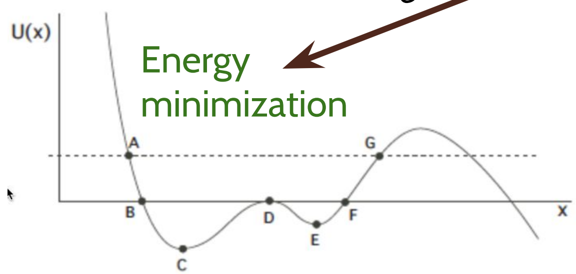
Monotonicity

Power flow solutions

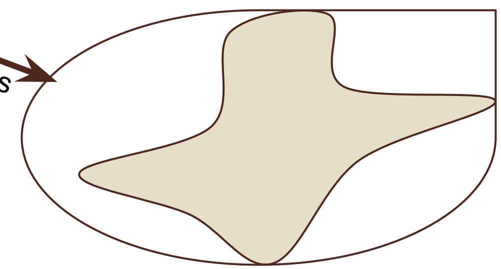
Stability

Existence & Uniqueness

EF: Energy minimization



CR: Convex relaxation



Theorem:

1. **(A)** Exists a power flow solution $\Longleftrightarrow$ **(B)** FP, EF and CR succeed
2. **All methods find the same (well-defined) high-voltage solution!**

Proof:

$\boxed{\text{FP solution}} \overset{\text{Maximality}}{=} \boxed{\text{CR solution}} \overset{\text{KKT + M-matrix theory}}{=} \boxed{\text{EF solution}}$

Properties of the solution

Let $v_{\mathrm{HV}} : \mathcal{S} \to \mathrm{R}^n$ be the map from power injections to the (unique) solution $v_{\mathrm{HV}}(p,q)$ given by FP, CR and EF.

A) $\mathcal{S}$ is **convex** and $v_{\mathrm{HV}}(p,q)$ is **continuous**

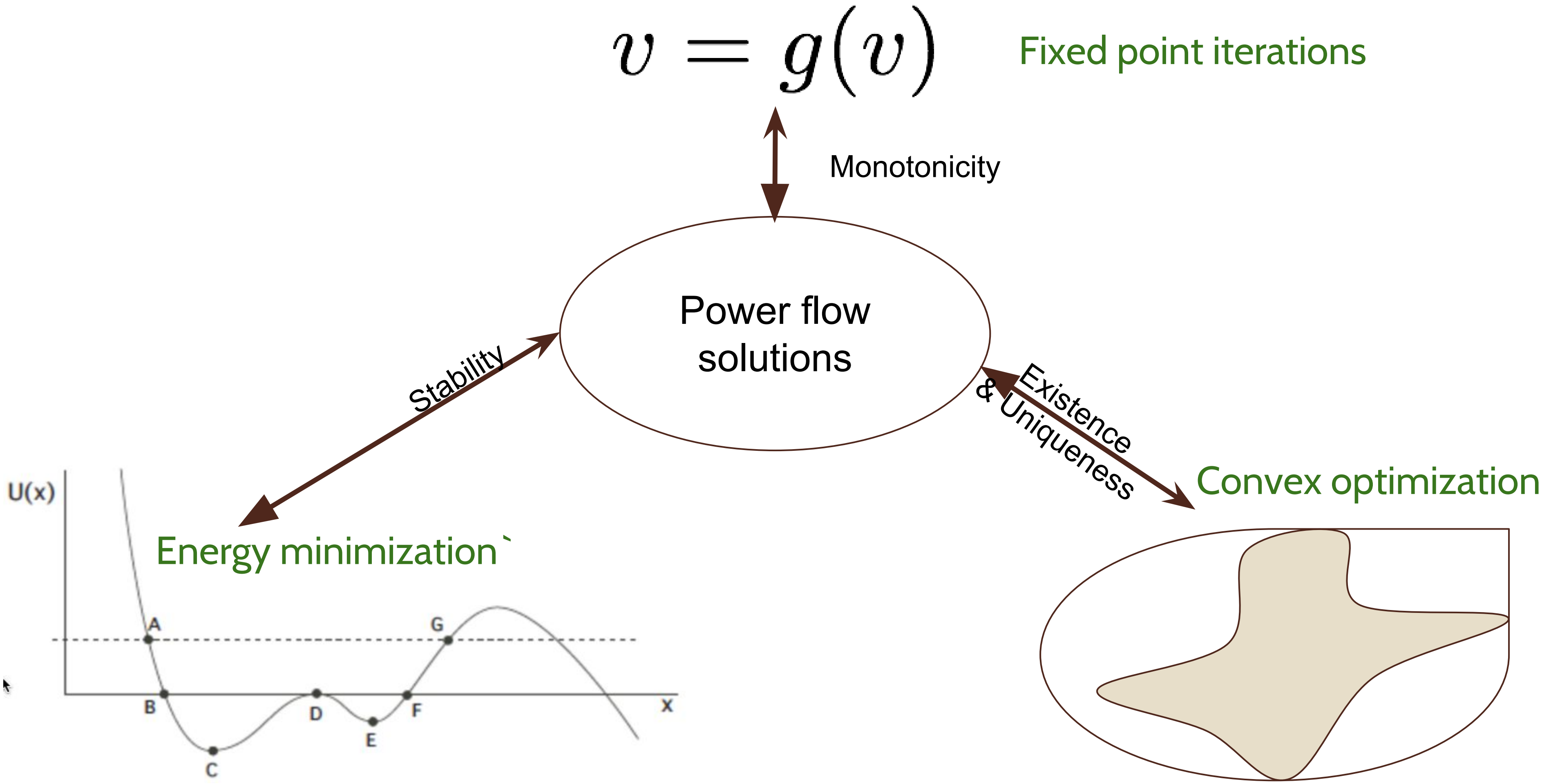
B) If $(p,q) \notin \mathcal{S}$, then no PF exists

C) **High-Voltage:** $v_{\mathrm{HV}}(p,q) \geq v'$ for any other PF solution of (p,q)

D) **Stability:** $(v_{\mathrm{HV}}, \theta_{\mathrm{HV}})(p,q)$ has positive definite Jacobian

E) **Voltage Regularity:** $\dfrac{\partial v_{\mathrm{HV}}}{\partial q} > \mathbf{0}_{n \times n}$ (element-wise)

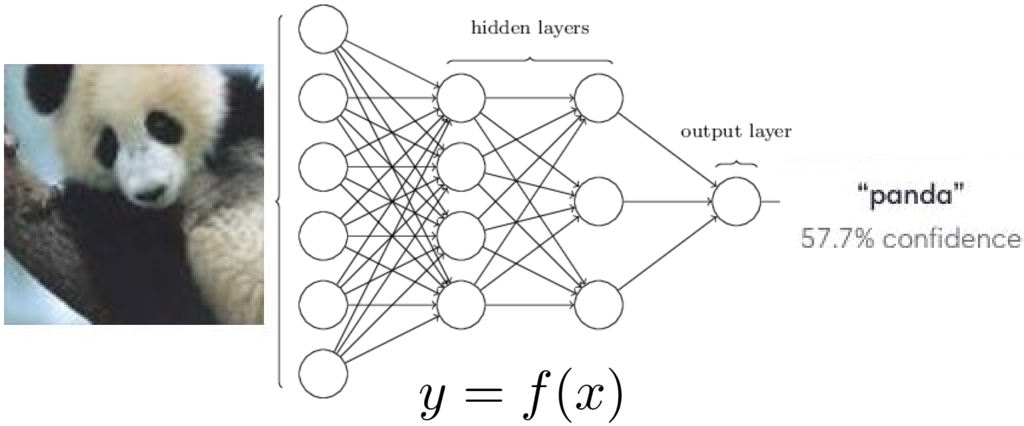
Summary



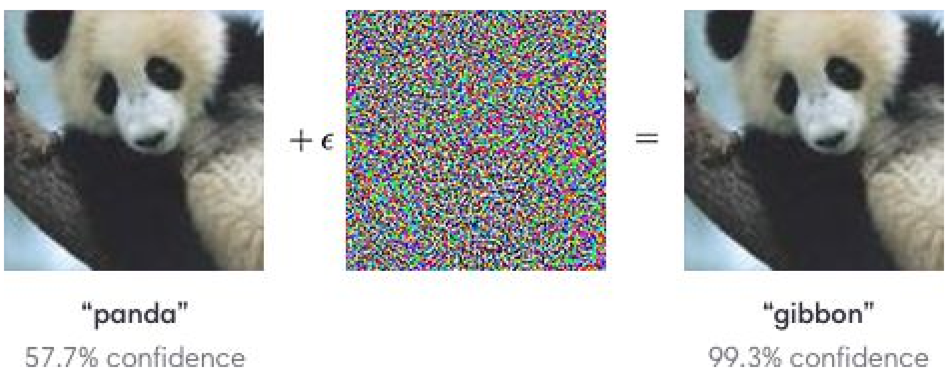
The **unique** common solution **is high-voltage**, **stable**, and **voltage regular**

…amenable for robust optimization!

Future outlook: Nonlinear Equations in deep learning



State of the art performance on complex image classification tasks [Krizhevsky et al, 2012]



But terribly fragile

Verifying neural networks $\equiv \nexists y \text{ s.t } f(x+y) \neq f(x), \|y\| \leq \delta$

Thanks!

Related Publication:

K Dvijotham, E Mallada, JW Simpson-Porco, “High-voltage solution in radial power networks: Existence, properties, and equivalent algorithms,” *IEEE Control Systems Letters*, 2017

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