Anglia Ruskin University | Global Sustainability Institute

Making Decisions Using Uncertain Forecasts

Environment Agency

Environmental Modelling in Industry Study Group, Cambridge – March 2017

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Rationale

- Probabilistic flood forecasting can provide a range of benefits when compared with conventional deterministic methods:

- Longer forecasting lead times

- Represents the inherent uncertainties

- Allows action to be taken earlier.

- However**, more information does not necessarily result in better decision-making,** particularly where the probabilistic forecasts contain conflicting predictions.

Challenge

Evaluate the (mis)use of probabilistic flood forecasts in incident response and proactive flood management

Routine decisions

i.e. issue a flood warning, closing a flood barrier, evacuation = least-cost optimisation

Reactive decisions

i.e. heuristics, lookup tables, risk appetite and bias = rules of thumb

Example

Colne Barrier, Exeter

Event	P4_H4	P4_H5	P4_H6	P4_H7	P5_H1	P5_H2	P5_H3
Date & Time	13/01/2009 13:45	14/01/2009 02:15	14/01/2009 14:30	15/01/2009 03:00	09/02/2009 12:00	10/02/2009 00:15	10/02/2009 12:30
Actural peak Water Level (mAOD)	3.293	2.973	3.021	2.717	2.978	3.070	3.482
Actual closure ?????							
Deterministic Flood Forecast							
Forecast peak water level (mAOD)	3.33	3.01	3.07	2.88	3.15	3.05	3.55
Closure Threshold (mAOD)	3.20	3.20	3.20	3.20	3.20	3.20	3.20
Closure Decision	Y	N	N	N	N	N	Y
Decision Cost (£)	£4,000	£0	£0	£0	£0	£0	£4,000
Decision Benefit (£)	£0	£0	£0	£0	£0	£0	£2,247,776
Hit	0	0	0	0	0	0	1
False Alarm	1	0	0	0	0	0	0
Miss	0	0	0	0	0	0	0
No Event	0	1	1	1	1	1	0

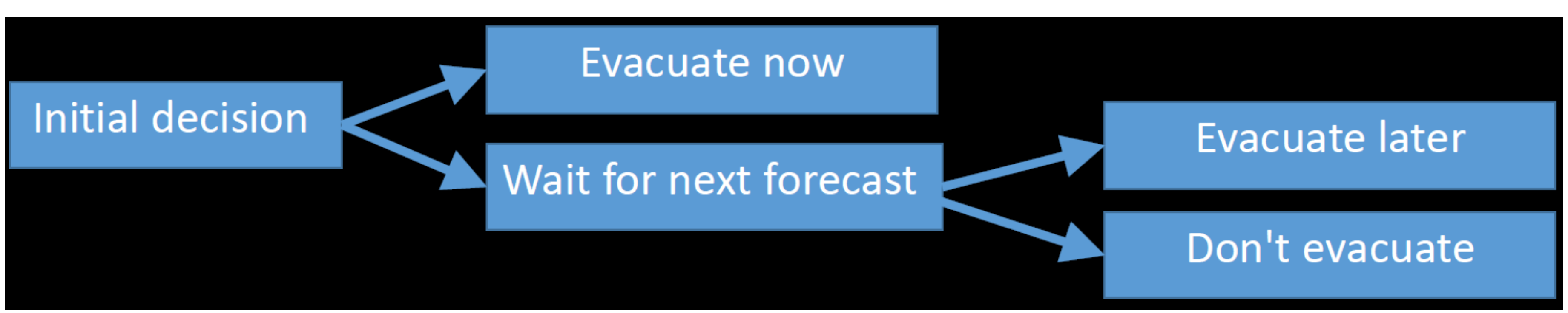
Probabilistic Flood Forecast	Peak Level (mAOD)	Closure Benefit (£)	Peak Level (mAOD)	Closure Benefit (£)	Peak Level (mAOD)	Closure Benefit (£)	Peak Level (mAOD)	Closure Benefit (£)	Peak Level (mAOD)	Closure Benefit (£)	Peak Level (mAOD)	Closure Benefit (£)	Peak Level (mAOD)	Closure Benefit (£)
Ensemble 1	3.432	£1,629,727	3.152	£0	3.192	£0	2.948	£0	3.033	£0	3.188	£0	3.691	£4,831,256
Ensemble 2	3.423	£1,519,064	3.156	£0	3.189	£0	2.921	£0	3.032	£0	3.184	£0	3.682	£4,714,939
Ensemble 3	3.431	£1,617,018	3.157	£0	3.192	£0	2.947	£0	3.032	£0	3.188	£0	3.748	£5,537,264
Ensemble 4	3.403	£1,272,846	3.142	£0	3.186	£0	2.943	£0	3.038	£0	3.195	£0	3.694	£4,870,774
Ensemble 5	3.438	£1,702,941	3.159	£0	3.187	£0	2.924	£0	3.054	£0	3.142	£0	3.675	£4,632,138
Ensemble 6	3.400	£1,233,626	3.149	£0	3.199	£0	2.951	£0	3.023	£0	3.183	£0	3.690	£4,812,196
Ensemble 7	3.428	£1,584,633	3.148	£0	3.193	£0	2.973	£0	3.038	£0	3.233	£0	3.677	£4,658,119
Ensemble 8	3.432	£1,625,081	3.147	£0	3.179	£0	2.957	£0	3.023	£0	3.220	£0	3.679	£4,678,587
Ensemble 9	3.421	£1,492,969	3.148	£0	3.192	£0	2.943	£0	3.024	£0	3.207	£0	3.665	£4,505,170
Ensemble 10	3.432	£1,633,428	3.148	£0	3.191	£0	2.949	£0	3.022	£0	3.149	£0	3.650	£4,323,338

(Problem framing) Challenge

Multiple forecast/multiple decisions

	No Control Barrier	Control Barrier	Partial Defence
Big flood 30%	6	10	10
Small flood 50%	4	3	2
No flood 20%	4	1	2

Branching ('wait and see') decisions



Challenge

Evaluate the (mis)use of probabilistic flood forecasts in incident response and proactive flood management

Routine Decisions

i.e. issue a flood warning, closing a flood barrier, evacuation = least-cost optimisation

Reactive decisions

i.e. heuristics, lookup tables, risk appetite and bias = rules of thumb

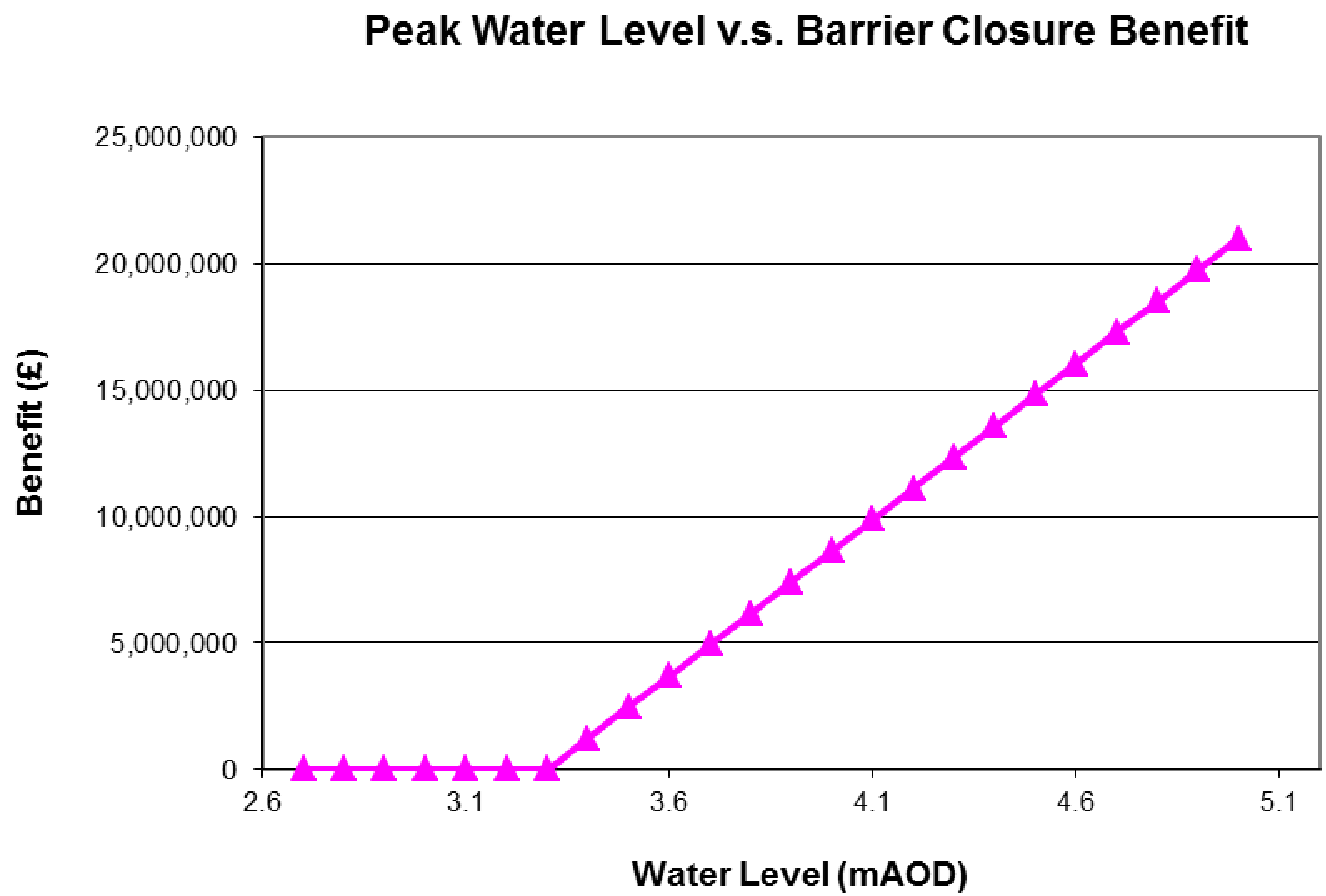
Objective:

Develop an easy-to-use decision making tool to be applied to multiple forecast, multiple action, delayed decisions.

Problem framing

	Evacuate	Control Barrier	Partial Defence	Water Course Clearing	No Action
Cost	50,000	100,000	30,000	15,000	0
Benefit					
Ensemble One	8,250,282	13,414,096	5.455638	2,727,819	0
Ensemble Two	4,825,375	7,755,312	3,192,124	1,596,062	0
…	…	…	…	…	…

Costing



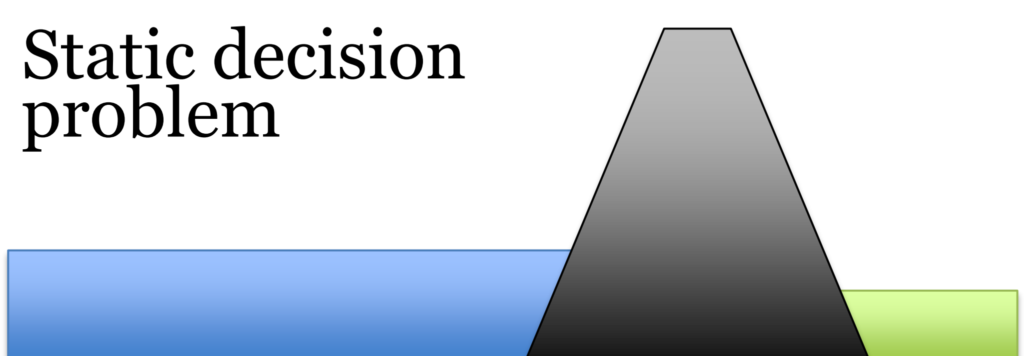
EA costing incorporates effect of different factors: social, risk to life, property damage…

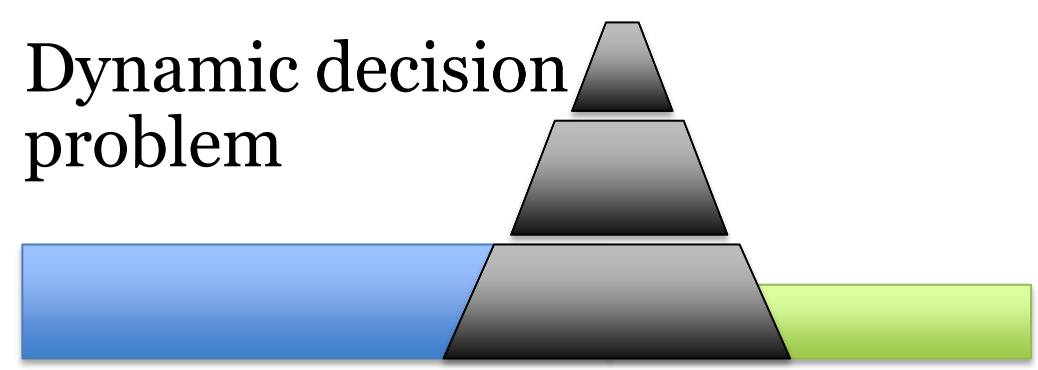
Implementation Assumption:

Other actions have a relative effect on each potential damage

Evaluation

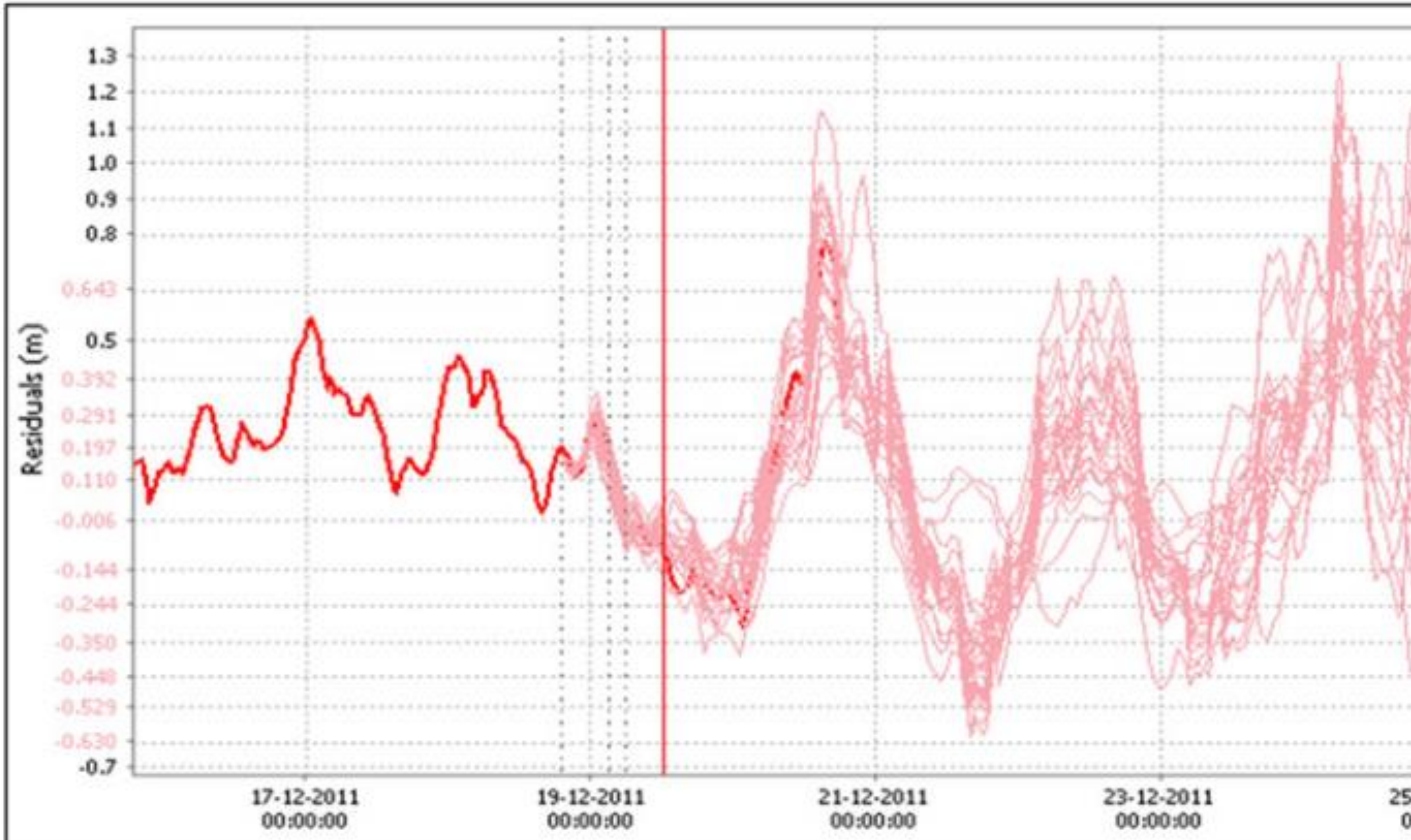
	Scenario				*Non-probabilistic decision criteria*				
Option	S1	S2	S3	etc.	Average (*Laplace*)	Minimum (*Maximin*)	Maximum (*Maximax*)	Minimum regret (*Minimax regret*)	Weighted average (*Hurwicz*)
A	10	20	50	100	45	10	100	900	55
B	2	3	3	1000	252✓	2	1000✓	199✓	501✓
C	200	200	202	202	201	200✓	202	798	201
D	100	110	120	410	185	100	410	590	255
etc.									
Best outcome	200	200	202	1000	Non-probabilistic decision outcome (✓)				
Best option	C	C	C	B	B	C	B	B	B



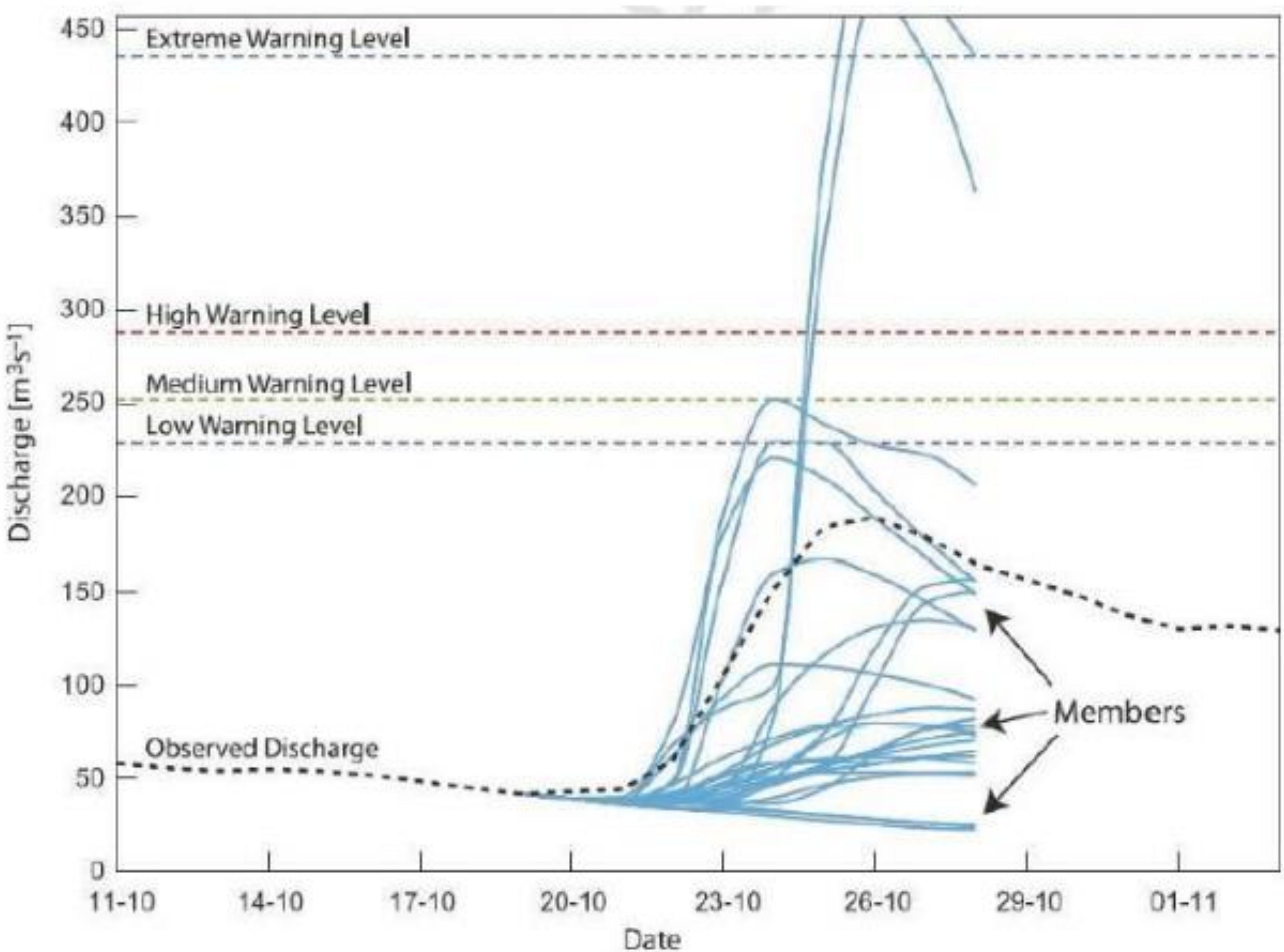


Making decisions with ~~probabilistic~~ uncertain forecasts

	Evacuate	Don't Evacuate	Temporary defences
Big flood 30%	6	10	10
Small flood 50%	4	3	2
No flood 20%	4	1	2



Residuals (m)
1.3
1.2
1.1
1.0
0.9
0.8
0.643
0.5
0.392
0.291
0.197
0.110
-0.006
-0.144
-0.244
-0.350
-0.440
-0.529
-0.630
-0.7
17-12-2011 00:00:00
19-12-2011 00:00:00
21-12-2011 00:00:00
23-12-2011 00:00:00



Extreme Warning Level
High Warning Level
Medium Warning Level
Low Warning Level
Observed Discharge
Members
Discharge [m3s-1]
Date
450
400
350
300
250
200
150
100
50
0
11-10
14-10
17-10
20-10
23-10
26-10
29-10
01-11

Probabilistic Forecast Data

	Level (mAOD)	Flood impact avoided by action (£)	Exceeding threshold?
Ensemble 1	3.297	£0	0
Ensemble 2	3.296	£0	0
Ensemble 3	3.264	£0	0
Ensemble 4	3.277	£0	0
Ensemble 5	3.317	£208,981	1
Ensemble 6	3.318	£224,816	1
Ensemble 7	3.285	£0	0
Ensemble 8	3.331	£386,912	1
Ensemble 9	3.330	£376,332	1
Ensemble 10	3.288	£0	0
Ensemble 11	3.291	£0	0
Ensemble 12	3.336	£442,730	1
Ensemble 13	3.297	£0	0
Ensemble 14	3.296	£0	0
Ensemble 15	3.264	£0	0
Ensemble 16	3.292	£0	0
Ensemble 17	3.302	£25,561	1
Ensemble 18	3.342	£513,820	1
Ensemble 19	3.292	£0	0
Ensemble 20	3.288	£0	0
Ensemble 21	3.310	£124,276	1
Ensemble 22	3.310	£124,032	1
Ensemble 23	3.272	£0	0
Ensemble 24	3.284	£0	0

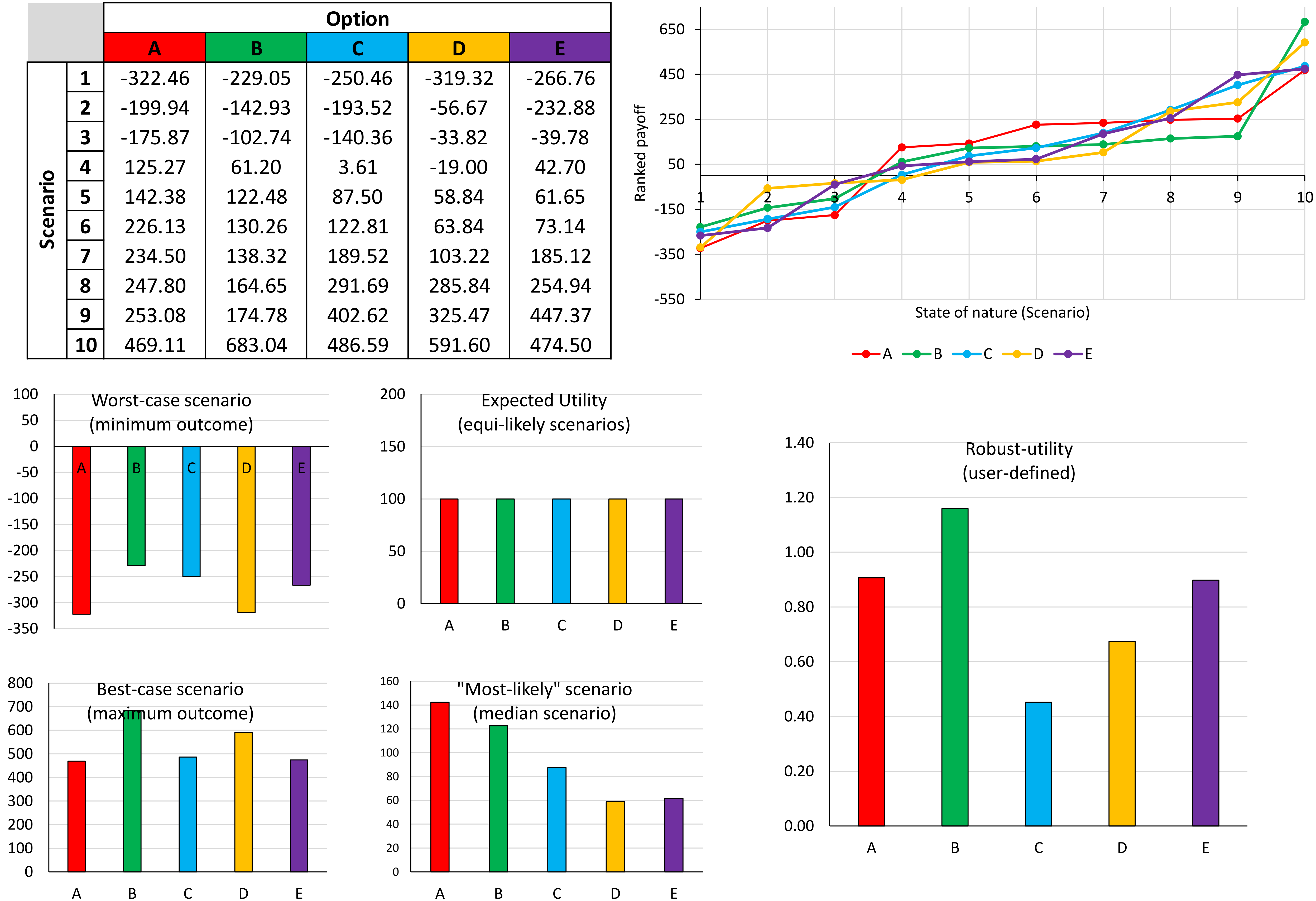
Expected Action Benefit (£)	£101,144	
Action Level Threshold (mAOD)		3.3
Exceeding probability		38%

	Event	P8_H2		P8_H3		P8_H4		P8_H5		P8_H6	
	Date & Time	31/01/2010 13:00		01/02/2010 01:15		01/02/2010 13:45		02/02/2010 02:15		02/02/2010 14:15	
5	Actural peak Water Level (mAOD)	3.109		3.063		3.201		2.913		3.223	
	Probabilistic Flood Forecast	**Peak Level (mAOD)**	**Closure Benefit (£)**	**Peak Level (mAOD)**	**Closure Benefit (£)**	**Peak Level (mAOD)**	**Closure Benefit (£)**	**Peak Level (mAOD)**	**Closure Benefit (£)**	**Peak Level (mAOD)**	**Closure Benefit (£)**
	Ensemble 1	3.454	£1,902,721	3.302	£22,598	3.513	£2,631,165	3.199	£0	3.335	£435,950
	Ensemble 2	3.448	£1,830,456	3.298	£0	3.502	£2,492,383	3.203	£0	3.306	£75,112
	Ensemble 3	3.447	£1,811,789	3.282	£0	3.509	£2,586,432	3.211	£0	3.313	£161,385
	Ensemble 4	3.451	£1,860,180	3.292	£0	3.522	£2,743,694	3.197	£0	3.310	£122,475
	Ensemble 5	3.445	£1,794,370	3.277	£0	3.499	£2,454,749	3.217	£0	3.345	£558,540
	Ensemble 6	3.458	£1,954,484	3.311	£136,910	3.502	£2,497,445	3.176	£0	3.330	£371,499
	Ensemble 7	3.439	£1,721,763	3.299	£0	3.501	£2,479,104	3.199	£0	3.282	£0
	Ensemble 8	3.462	£1,999,359	3.303	£36,462	3.518	£2,692,539	3.193	£0	3.319	£238,502
	Ensemble 9	3.458	£1,952,591	3.299	£0	3.515	£2,659,119	3.222	£0	3.343	£531,224
	Ensemble 10	3.454	£1,905,899	3.288	£0	3.524	£2,761,995	3.186	£0	3.333	£406,898
	Ensemble 11	3.455	£1,908,725	3.306	£68,629	3.521	£2,723,441	3.178	£0	3.361	£755,255
	Ensemble 12	3.453	£1,890,068	3.310	£121,119	3.510	£2,591,980	3.199	£0	3.380	£992,322
	Ensemble 13	3.454	£1,906,360	3.312	£145,650	3.520	£2,722,821	3.175	£0	3.341	£509,867
	Ensemble 14	3.448	£1,831,537	3.289	£0	3.504	£2,521,741	3.192	£0	3.354	£661,074
	Ensemble 15	3.464	£2,020,999	3.306	£74,736	3.520	£2,719,812	3.223	£0	3.337	£458,360
	Ensemble 16	3.449	£1,836,213	3.301	£16,772	3.514	£2,639,421	3.197	£0	3.287	£0
	Ensemble 17	3.467	£2,060,485	3.307	£91,073	3.508	£2,570,101	3.190	£0	3.349	£601,623
	Ensemble 18	3.457	£1,945,171	3.318	£221,028	3.505	£2,526,135	3.212	£0	3.344	£543,720
	Ensemble 19	3.460	£1,981,190	3.300	£0	3.508	£2,573,999	3.189	£0	3.375	£920,476
	Ensemble 20	3.463	£2,012,224	3.300	£0	3.518	£2,687,949	3.199	£0	3.343	£532,226
	Ensemble 21	3.455	£1,913,867	3.305	£57,575	3.506	£2,541,207	3.178	£0	3.362	£771,086
	Ensemble 22	3.464	£2,027,166	3.312	£149,596	3.509	£2,577,662	3.217	£0	3.353	£654,154
	Ensemble 23	3.441	£1,741,614	3.314	£167,003	3.513	£2,627,437	3.176	£0	3.347	£576,720
	Ensemble 24	3.453	£1,889,914	3.302	£19,179	3.501	£2,480,560	3.214	£0	3.282	£0
	Expected Closure Benefit (£)	£1,904,131		£55,347		£2,604,287		£0		£453,270	
	Closure Cost (£)	£4,000		£4,000		£4,000		£4,000		£4,000	
21	Closure decision	Y		Y		Y		N		Y	
5	Hit	0		0		0		0		0	
16	False Alarm	1		1		1		0		1	
0	Miss	0		0		0		0		0	
52	No Event	0		0		0		1		0	

Operation Panel | **Performance Assessment** | MCA_Results | Sheet1

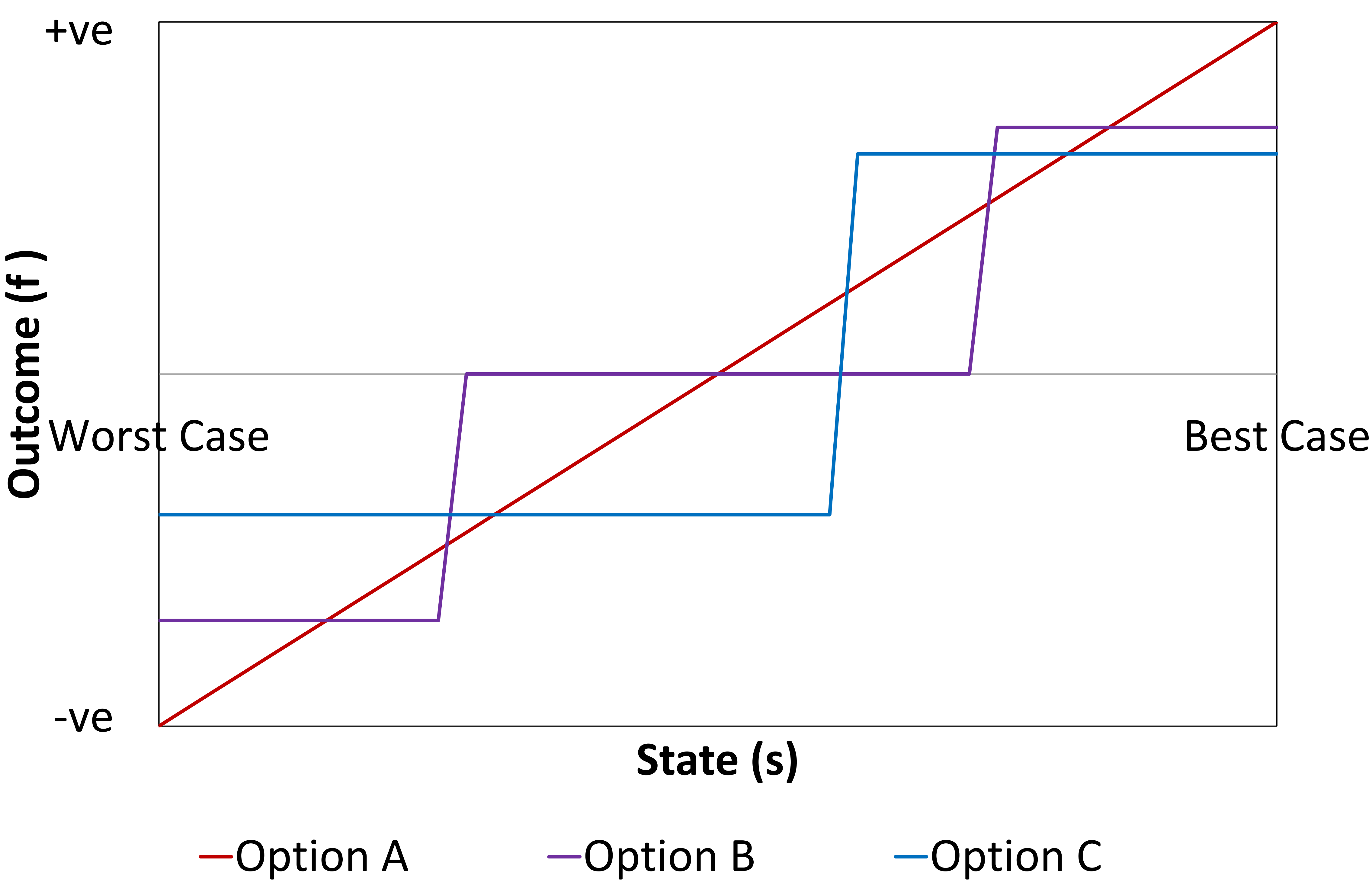
Synthetic data

Scenario	A	B	C	D	E
1	-322.46	-229.05	-250.46	-319.32	-266.76
2	-199.94	-142.93	-193.52	-56.67	-232.88
3	-175.87	-102.74	-140.36	-33.82	-39.78
4	125.27	61.20	3.61	-19.00	42.70
5	142.38	122.48	87.50	58.84	61.65
6	226.13	130.26	122.81	63.84	73.14
7	234.50	138.32	189.52	103.22	185.12
8	247.80	164.65	291.69	285.84	254.94
9	253.08	174.78	402.62	325.47	447.37
10	469.11	683.04	486.59	591.60	474.50



Try it yourself

Q) Do you prefer Option A, B or C?



Robust-utility

$$z = \max_{d \epsilon D}\left((\alpha. A) - ((1-\alpha). B)\right) d$$

$$A = \left(\frac{(a)d - \min_{d\epsilon D}(a)d}{\max_{d\epsilon D}(a)d - \min_{d\epsilon D}(a)d}\right)$$

$$B = \left(\frac{(b)d - \min_{d\epsilon D}(b)d}{\max_{d\epsilon D}(b)d - \min_{d\epsilon D}(b)\, d}\right)$$

$$a = \sum_{s=1}^{n}\left(\frac{((f)s - \chi)}{\left(\max_{s\epsilon S}(f)s - \chi\right)}\right)$$

$$b = \sum_{s=1}^{n}\left(\frac{((f)s - t)}{\left(\min_{s\epsilon S}(f)s - t\right)}\right)$$

$$\chi = \left(\max_{s=n} f - \left(\left(\max_{s\epsilon S}(f)s - \min_{s\epsilon S}(f)s\right).\left(\frac{\beta}{100}\right)\right)\right)$$

Where

z = decision outcome
d = option/s
α = coefficient of optimism (0-1)
f = outcome
n = number of states
β = coefficient of robustness (0-100)
t = threshold (e.g. 0)
s = state

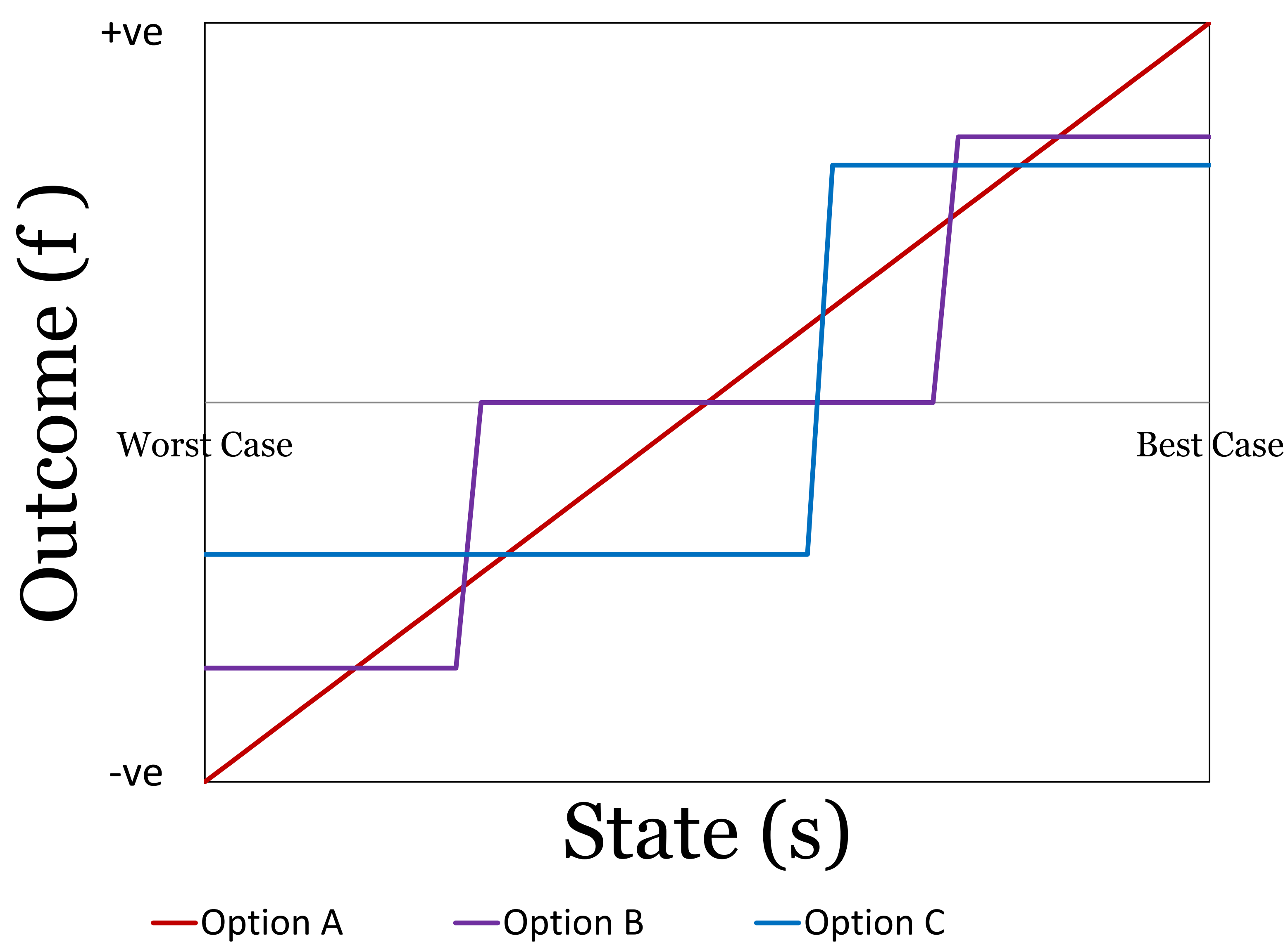
Advantages:

- Exploratory decision tool
- Accommodate a range of risk appetites
- Incorporate threshold concepts
- Supports static and adaptive decision making
- Does not rely on probabilities
- Highly reproducible from small sub samples
- Can be easily integrated with more advanced techniques
- Easy to implement

Green and Weatherhead, 2014

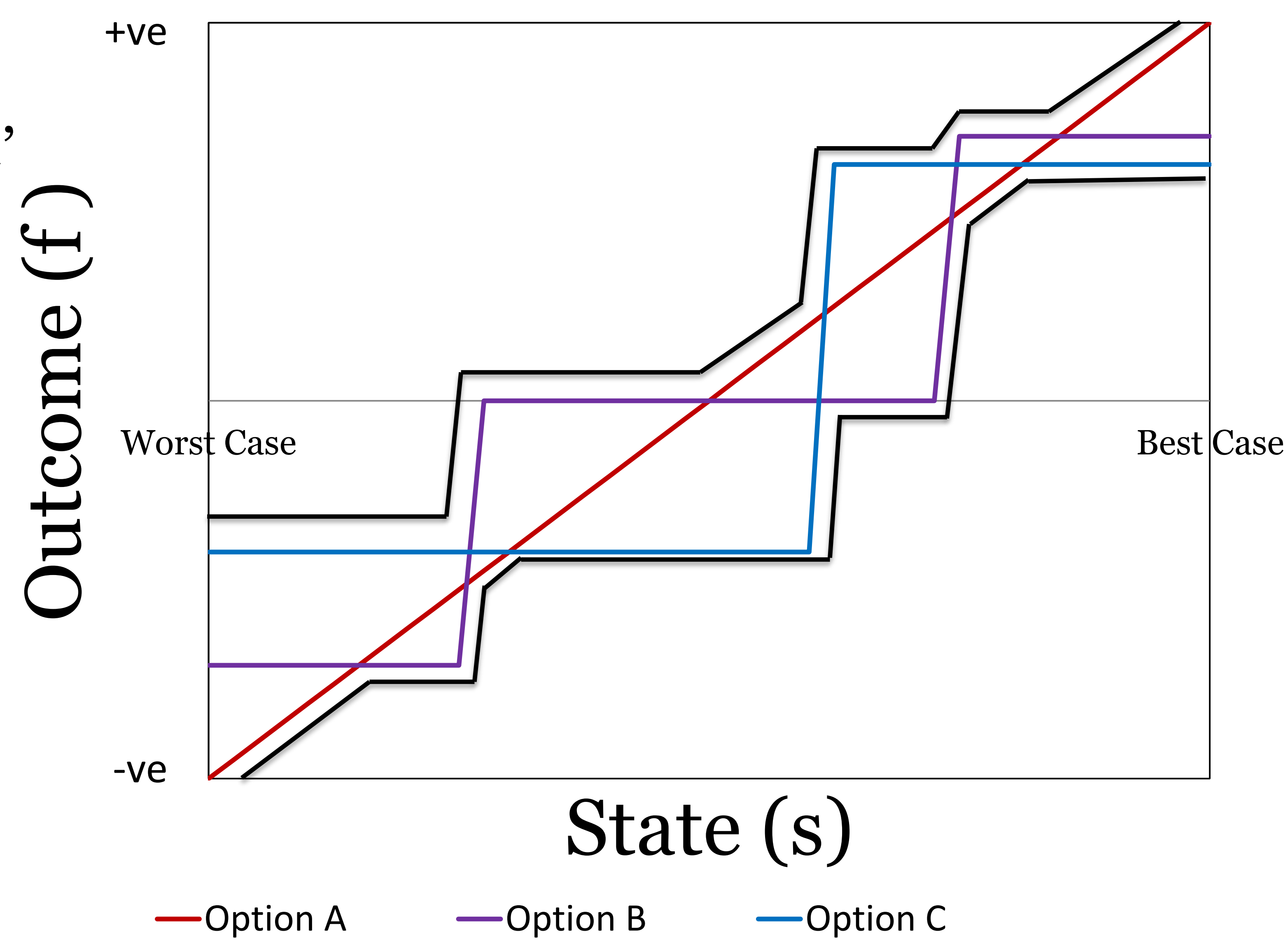
Robust-utility

- Plot the pay-off of the action against each scenario



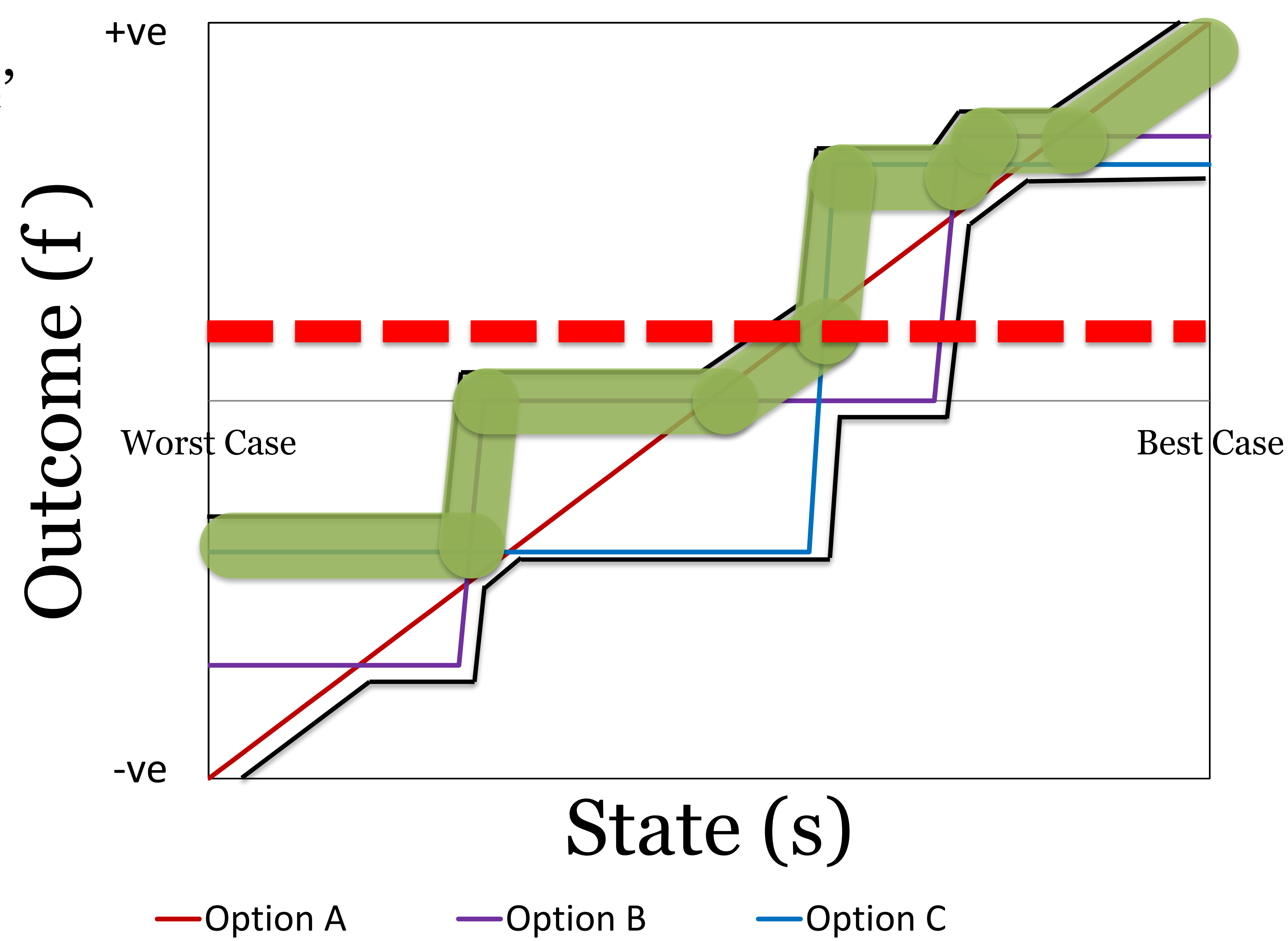
Robust-utility

- Plot the pay-off of the action against each scenario
- Identify 'best-possible' & 'worst possible' outcome



Robust-utility

- Plot the pay-off of the action against each scenario
- Identify 'best-possible' & 'worst possible' outcome
- Specify:
    - Robustness range
    - Threshold
    - Weighting coefficient
    - Score each option



So...

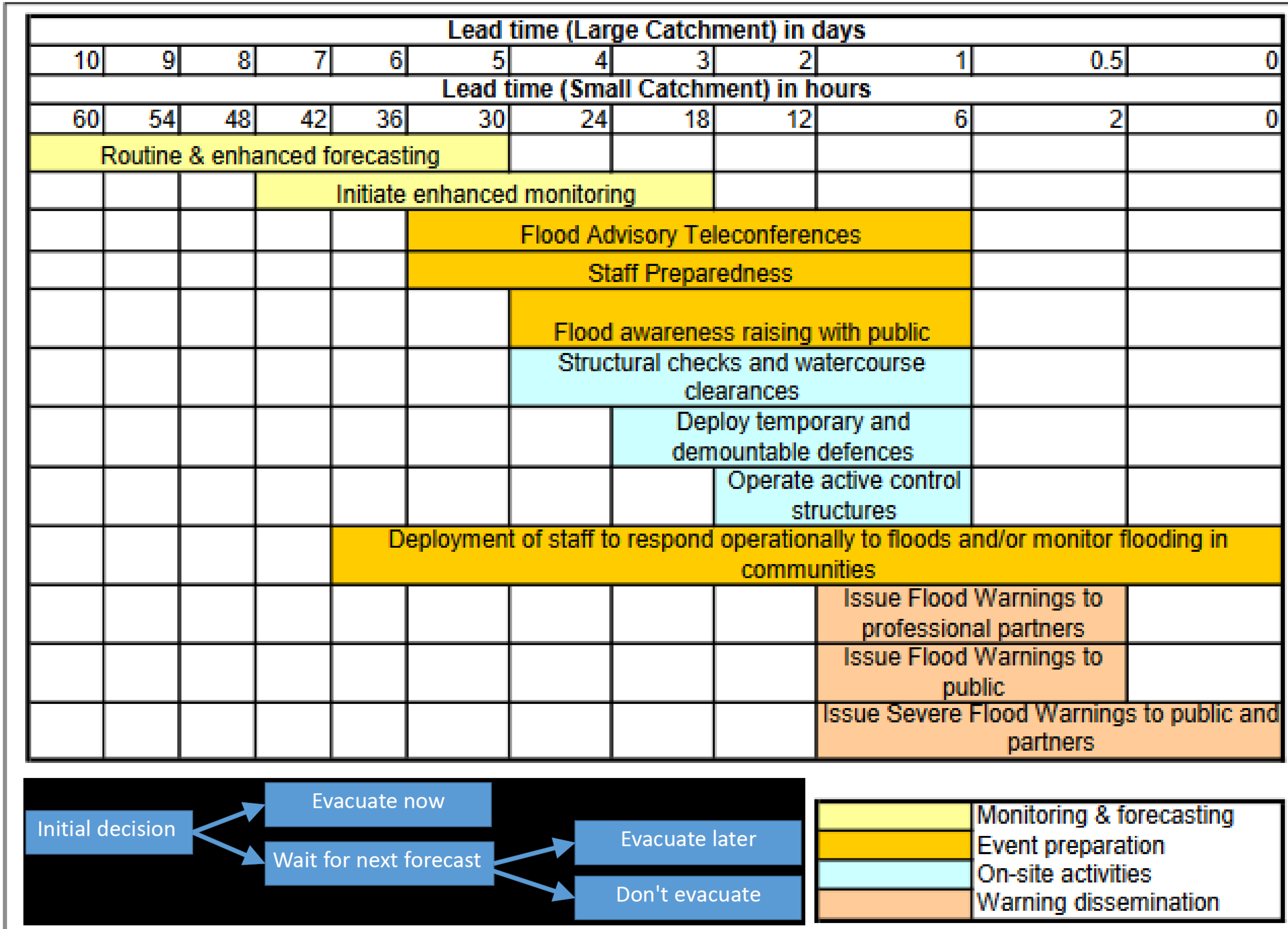
Coefficient of optimism (α)	Coefficient of robustness (β)	Threshold of acceptability (t)
0.5	80	0

State	f_d	$\max_{d\in D} f_d$	$\min_{d\in D} f_d$	χ	$(f_d - \chi)$	$(\max_{d\in D} f_d - \chi)$	$\left(\frac{(f_d-\chi)}{(\max_{d\in D} f_d-\chi)}\right)$
1	−10	−3	−15	−12.60	2.60	9.60	0.27
2	−8	−3	−15	−12.60	4.60	9.60	0.48
3	−6	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
4	−4	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
5	−2	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
6	0	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
7	2	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
8	4	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
9	6	15	3.6	5.88	0.12	9.12	0.01
10	8	15	3.6	5.88	2.12	9.12	0.23
11	10	15	3.6	5.88	4.12	9.12	0.45
Total							4.75

* This value is not calculated because $f_d < \chi$.

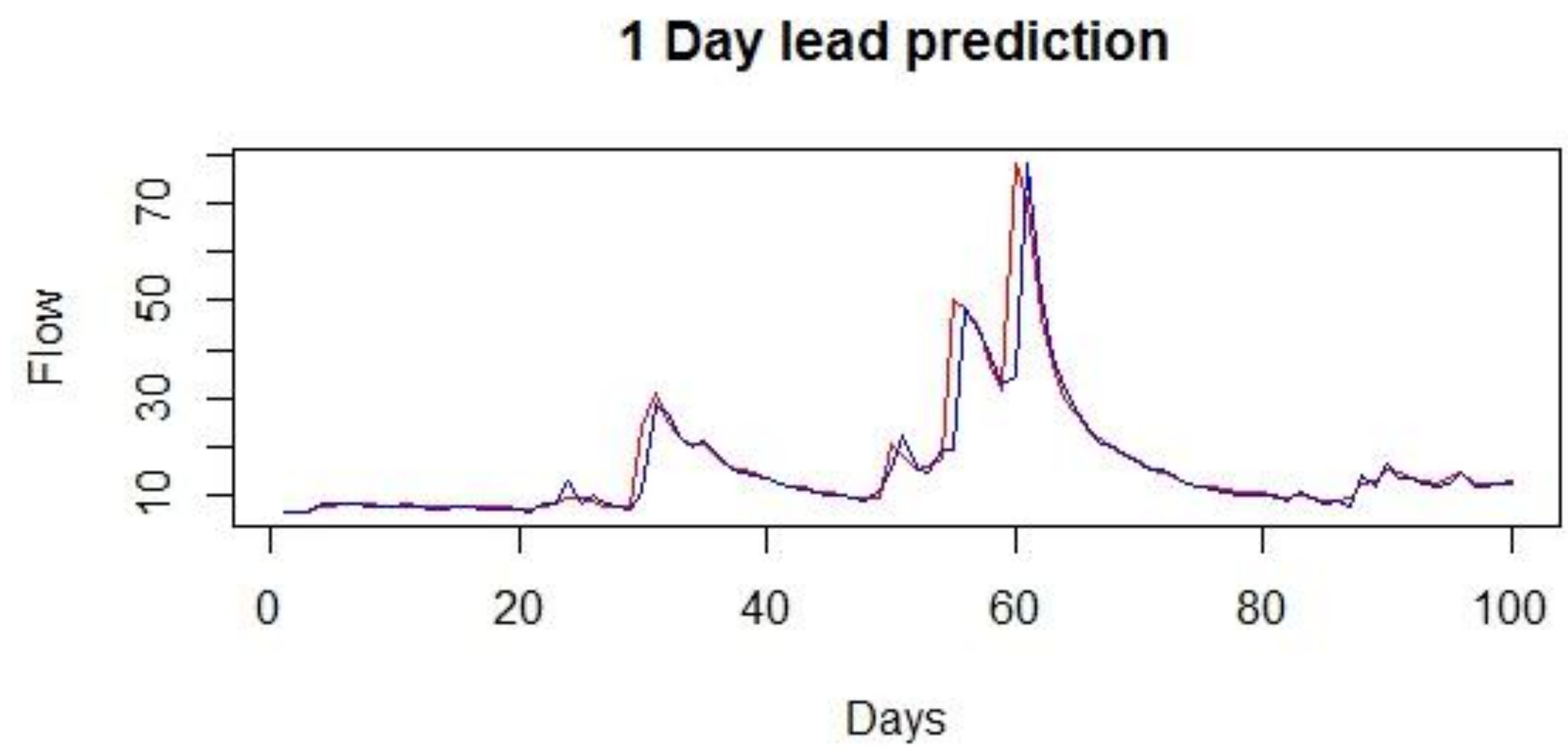
Option	A	B	$(\alpha \cdot A)$	$((1-\alpha)\cdot B)$	Green Z-score
A	4.75	3.27	2.37	1.63	0.74
B	6.00	3.00	3.00	1.50	1.50
C	4.94	3.35	2.47	1.68	0.79

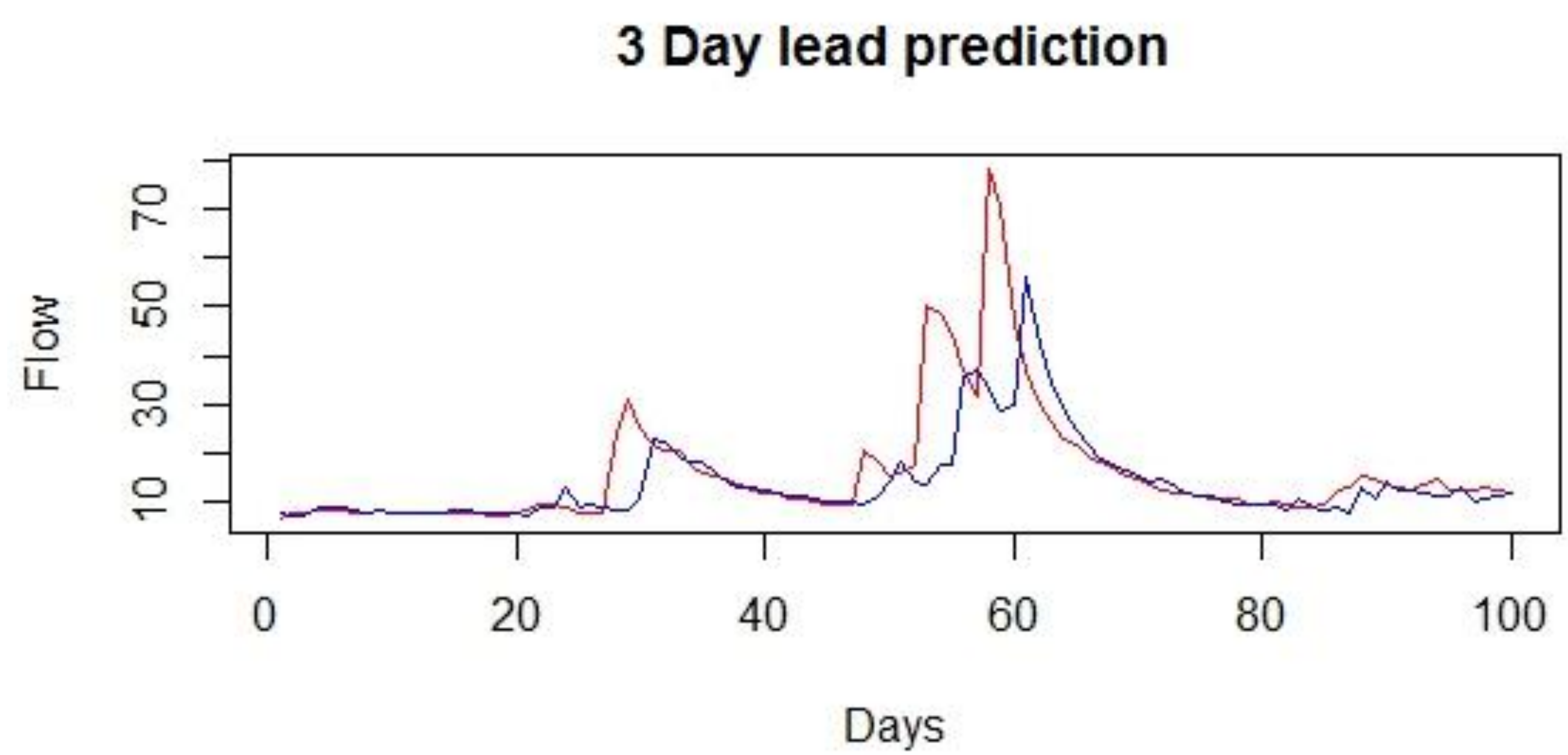
—Option A —Option B —Option C



Lead time (Large Catchment) in days
10
9
8
7
6
5
4
3
2
1
0.5
0
Lead time (Small Catchment) in hours
60
54
48
42
36
30
24
18
12
6
2
0
Routine & enhanced forecasting
Initiate enhanced monitoring
Flood Advisory Teleconferences
Staff Preparedness
Flood awareness raising with public
Structural checks and watercourse clearances
Deploy temporary and demountable defences
Operate active control structures
Deployment of staff to respond operationally to floods and/or monitor flooding in communities
Issue Flood Warnings to professional partners
Issue Flood Warnings to public
Issue Severe Flood Warnings to public and partners
Initial decision
Evacuate now
Wait for next forecast
Evacuate later
Don't evacuate
Monitoring & forecasting
Event preparation
On-site activities
Warning dissemination

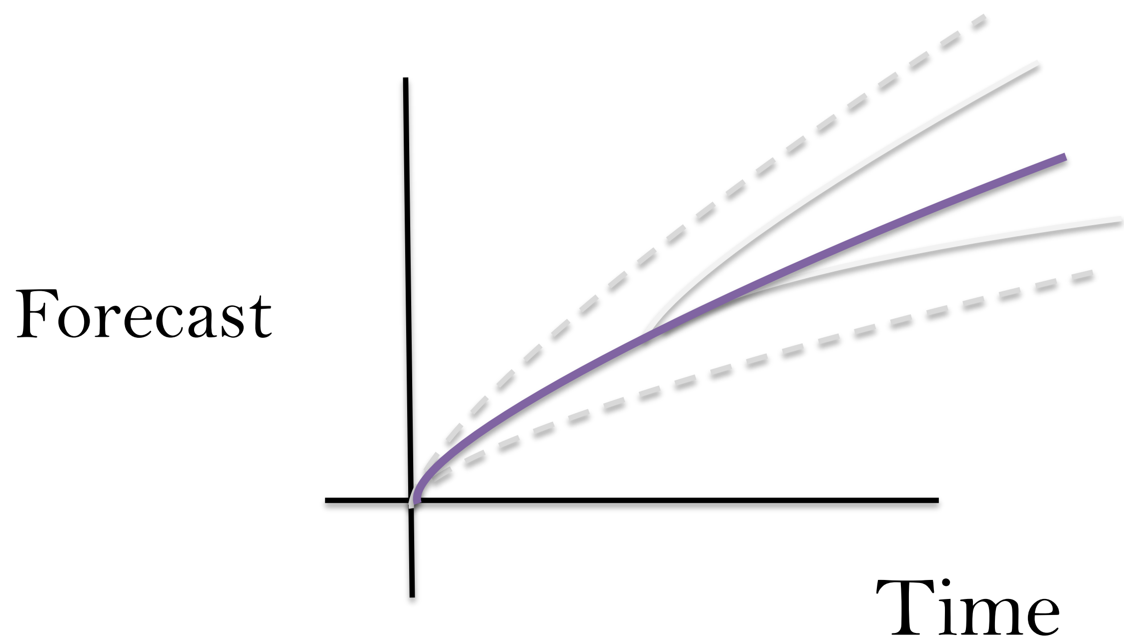
Credibility and delaying decisions





Credibility and delaying decisions

Question to answer: ***How*** *does the credibility of predictions change as we get closer to the predicted event and what* ***impact*** *does this have on decisions?*



Option One: Use historical data to calculate the expected cost of bad decisions

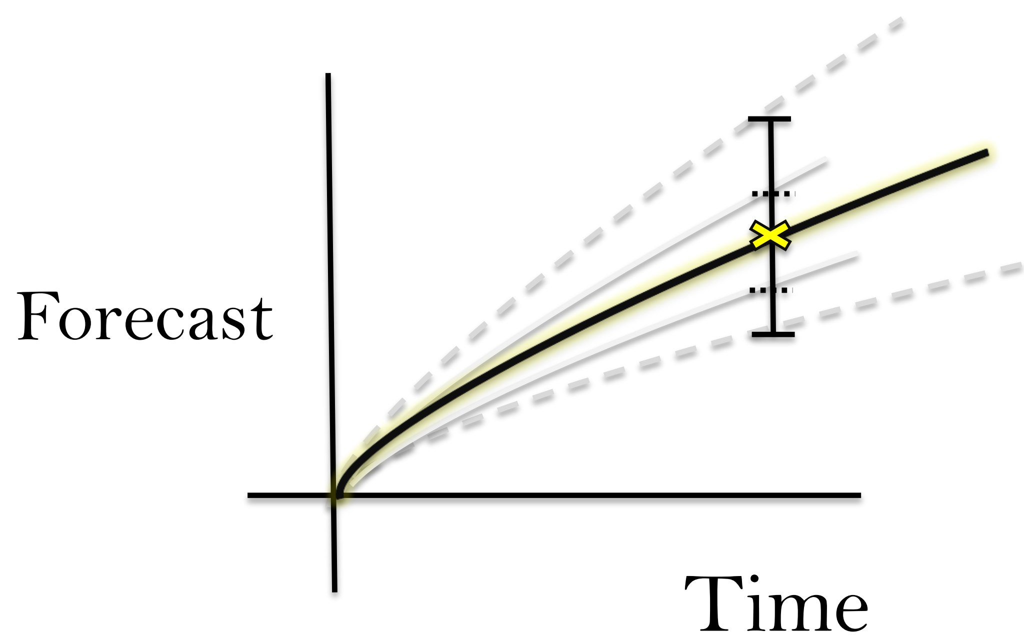
Note: *This relies on data existing and could be costly to run for each decision*

Credibility

Our proposal:

‘Relative Reliability Score’ to provide an ‘error fan’ around the prediction

Calculate whether decision would change at either end of the fan → Calculate whether decision would change with a smaller → Compare the relative associated costs



Construction of Decision Matrix

Costs of Mitigation actions: C_i

Expected damage caused per flood depth: $f(h)$

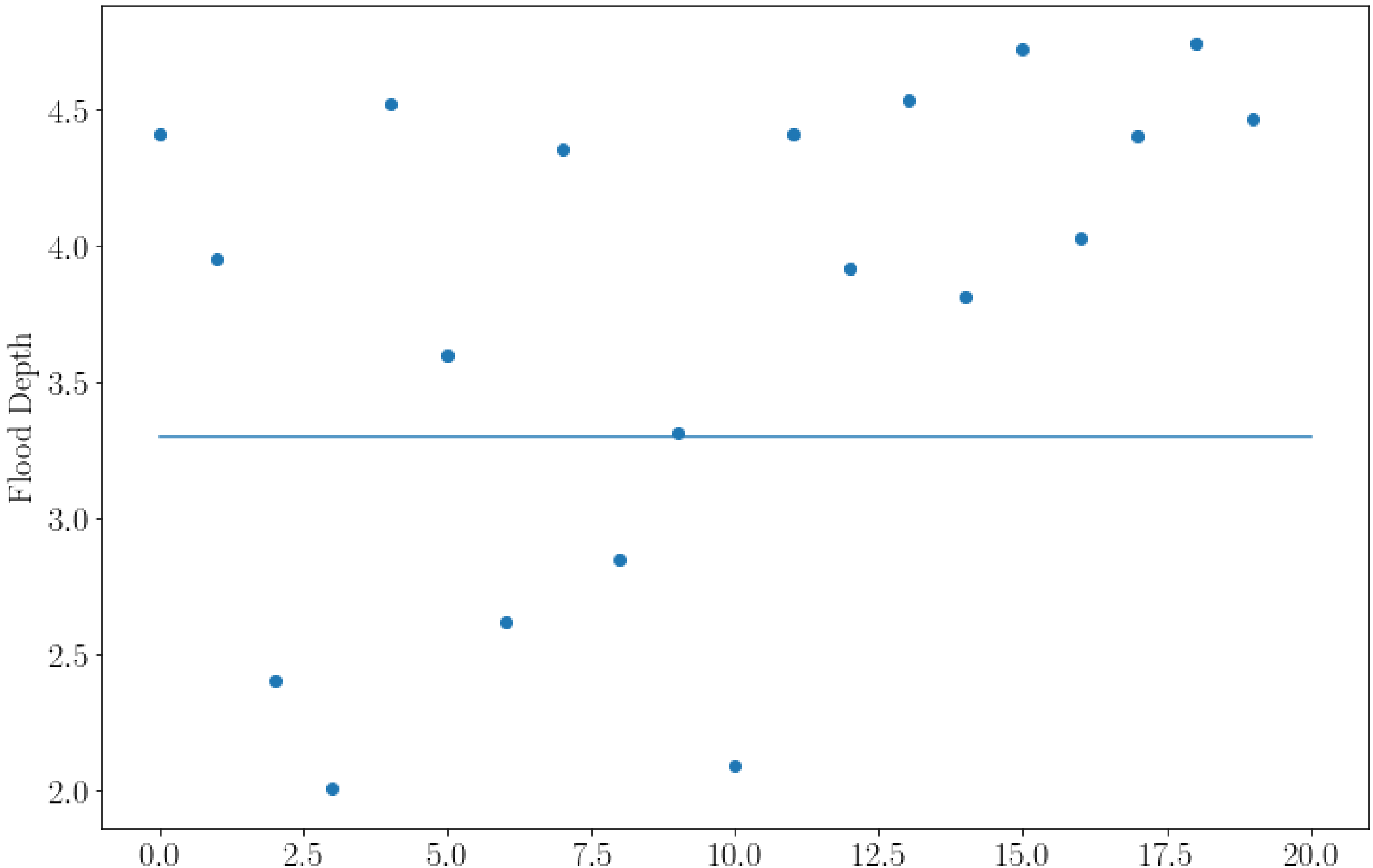
Ensemble Predictions of flood depths: h_{flood}

Avoided costs (benefit) of Mitigation actions: E_j

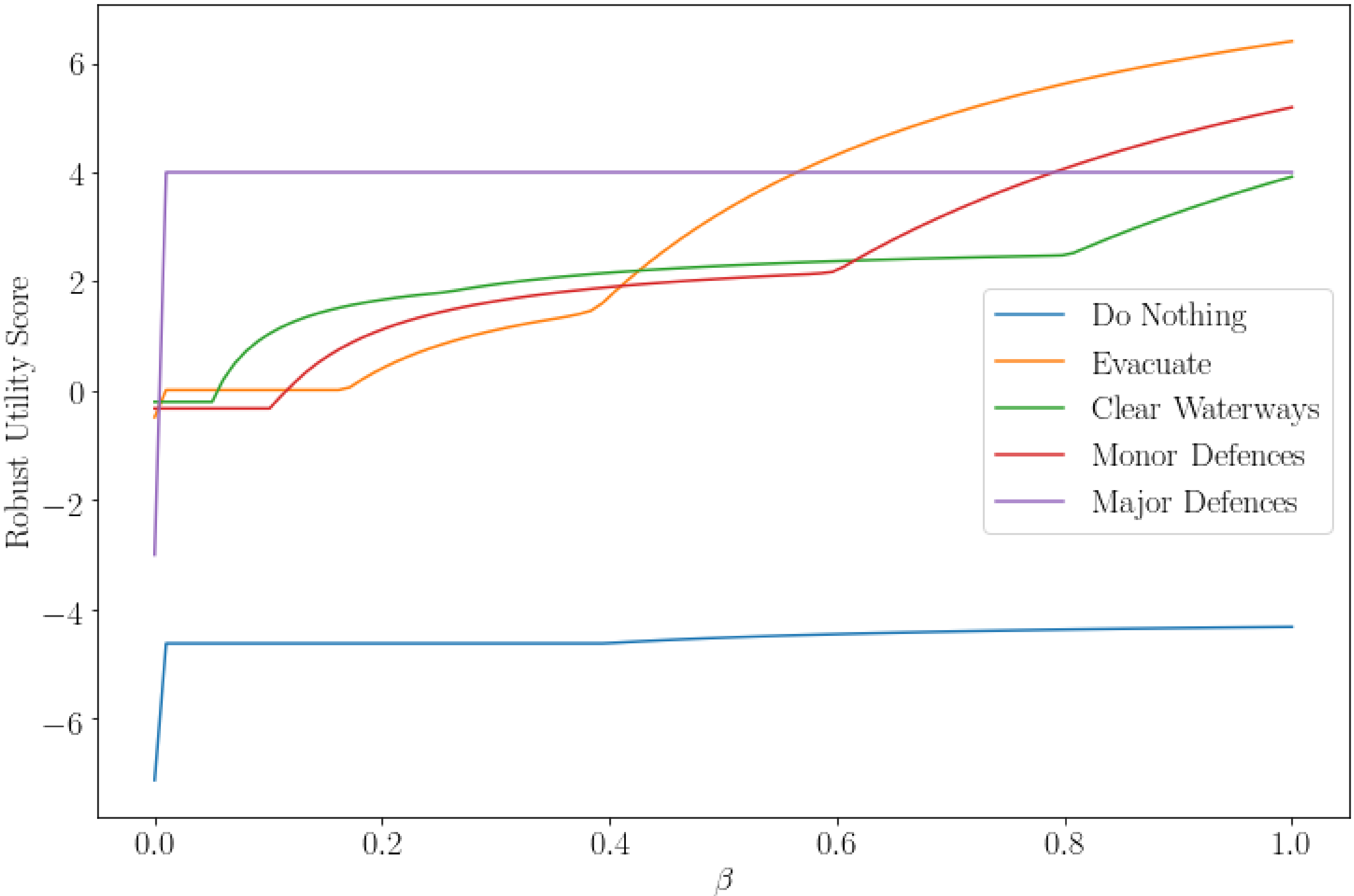
Decision Matrix

$$D_{ij} = C_i + E_j f(h_{flood})$$

Sample Ensembles



Robust Utility Scores



	Event	P8_H2		P8_H3		P8_H4		P8_H5		P8_H6	
	Date & Time	31/01/2010 13:00		01/02/2010 01:15		01/02/2010 13:45		02/02/2010 02:15		02/02/2010 14:15	
5	Actural peak Water Level (mAOD)	3.109		3.063		3.201		2.913		3.223	
	Probabilistic Flood Forecast	**Peak Level (mAOD)**	**Closure Benefit (£)**	**Peak Level (mAOD)**	**Closure Benefit (£)**	**Peak Level (mAOD)**	**Closure Benefit (£)**	**Peak Level (mAOD)**	**Closure Benefit (£)**	**Peak Level (mAOD)**	**Closure Benefit (£)**
	Ensemble 1	3.454	£1,902,721	3.302	£22,598	3.513	£2,631,165	3.199	£0	3.335	£435,950
	Ensemble 2	3.448	£1,830,456	3.298	£0	3.502	£2,492,383	3.203	£0	3.306	£75,112
	Ensemble 3	3.447	£1,811,789	3.282	£0	3.509	£2,586,432	3.211	£0	3.313	£161,385
	Ensemble 4	3.451	£1,860,180	3.292	£0	3.522	£2,743,694	3.197	£0	3.310	£122,475
	Ensemble 5	3.445	£1,794,370	3.277	£0	3.499	£2,454,749	3.217	£0	3.345	£558,540
	Ensemble 6	3.458	£1,954,484	3.311	£136,910	3.502	£2,497,445	3.176	£0	3.330	£371,499
	Ensemble 7	3.439	£1,721,763	3.299	£0	3.501	£2,479,104	3.199	£0	3.282	£0
	Ensemble 8	3.462	£1,999,359	3.303	£36,462	3.518	£2,692,539	3.193	£0	3.319	£238,502
	Ensemble 9	3.458	£1,952,591	3.299	£0	3.515	£2,659,119	3.222	£0	3.343	£531,224
	Ensemble 10	3.454	£1,905,899	3.288	£0	3.524	£2,761,995	3.186	£0	3.333	£406,898
	Ensemble 11	3.455	£1,908,725	3.306	£68,629	3.521	£2,723,441	3.178	£0	3.361	£755,255
	Ensemble 12	3.453	£1,890,068	3.310	£121,119	3.510	£2,591,980	3.199	£0	3.380	£992,322
	Ensemble 13	3.454	£1,906,360	3.312	£145,650	3.520	£2,722,821	3.175	£0	3.341	£509,867
	Ensemble 14	3.448	£1,831,537	3.289	£0	3.504	£2,521,741	3.192	£0	3.354	£661,074
	Ensemble 15	3.464	£2,020,999	3.306	£74,736	3.520	£2,719,812	3.223	£0	3.337	£458,360
	Ensemble 16	3.449	£1,836,213	3.301	£16,772	3.514	£2,639,421	3.197	£0	3.287	£0
	Ensemble 17	3.467	£2,060,485	3.307	£91,073	3.508	£2,570,101	3.190	£0	3.349	£601,623
	Ensemble 18	3.457	£1,945,171	3.318	£221,028	3.505	£2,526,135	3.212	£0	3.344	£543,720
	Ensemble 19	3.460	£1,981,190	3.300	£0	3.508	£2,573,999	3.189	£0	3.375	£920,476
	Ensemble 20	3.463	£2,012,224	3.300	£0	3.518	£2,687,949	3.199	£0	3.343	£532,226
	Ensemble 21	3.455	£1,913,867	3.305	£57,575	3.506	£2,541,207	3.178	£0	3.362	£771,086
	Ensemble 22	3.464	£2,027,166	3.312	£149,596	3.509	£2,577,662	3.217	£0	3.353	£654,154
	Ensemble 23	3.441	£1,741,614	3.314	£167,003	3.513	£2,627,437	3.176	£0	3.347	£576,720
	Ensemble 24	3.453	£1,889,914	3.302	£19,179	3.501	£2,480,560	3.214	£0	3.282	£0
	Expected Closure Benefit (£)	£1,904,131		£55,347		£2,604,287		£0		£453,270	
	Closure Cost (£)	£4,000		£4,000		£4,000		£4,000		£4,000	
21	Closure decision	Y		Y		Y		N		Y	
5	Hit	0		0		0		0		0	
16	False Alarm	1		1		1		0		1	
0	Miss	0		0		0		0		0	
52	No Event	0		0		0		1		0	

Operation Panel | **Performance Assessment** | MCA_Results | Sheet1

Output

Spreadsheet:

Input: Decision matrix, scenario predictions

Output: Robustness scores of decision and best decision

Python:

Randomly Generated Water Levels, actions determined by estimated reduction of damage

Input: α, β, t and Decision matrix

Output: Robustness scores of decisions, and best decision.

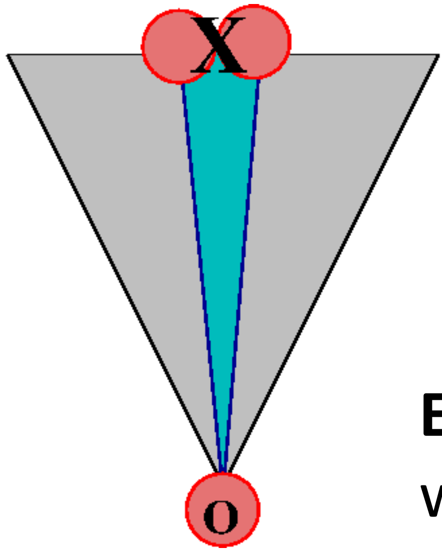
Further Work

Run with real life data and integrate to EA operations

Test using historic data to fine tune parameters

Implement a robust method for making decisions about delaying, using existing credibility information for forecasts.

Prediction probability vs. lead time

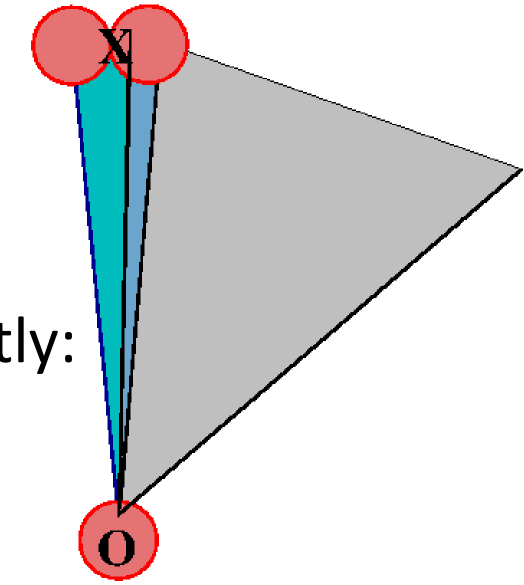


Crude estimate, 1

For typical impacts, $p \sim \frac{2L/R}{2\delta_\theta} = \frac{L}{R\,\delta_\theta} = \frac{L}{u\,t\,\delta_\theta} \sim \frac{1}{t}$

L storm size , $2\delta_\theta$ forecast cone width, u typical speed.

It is less than that for oblique impacts



Estimate, 2. Suppose center of the storm q is moving with an average speed u but direction is randomly rotated slightly:

$$\ddot{q} = \omega \times \dot{q}, \qquad \langle \int_t^{t+\tau} \omega^n \, dt \rangle = \mu_n \tau, n = 1,2$$

Probability density function in phase space for the storm center will satisfy eq. of the sort:

$$\frac{\partial f}{\partial t} = -p_i \frac{\partial f}{\partial q_i} + \kappa \left(-p_i \frac{\partial f}{\partial p_i} + p_1^2 \frac{\partial^2 f}{\partial^2 p_2} + p_2^2 \frac{\partial^2 f}{\partial^2 p_1} - 2\, p_1 p_2 \frac{\partial^2 f}{\partial p_1 \partial p_2} \right) = -p_i \frac{\partial f}{\partial q_i} + \kappa \frac{\partial^2 f}{\partial^2 \theta_p}$$

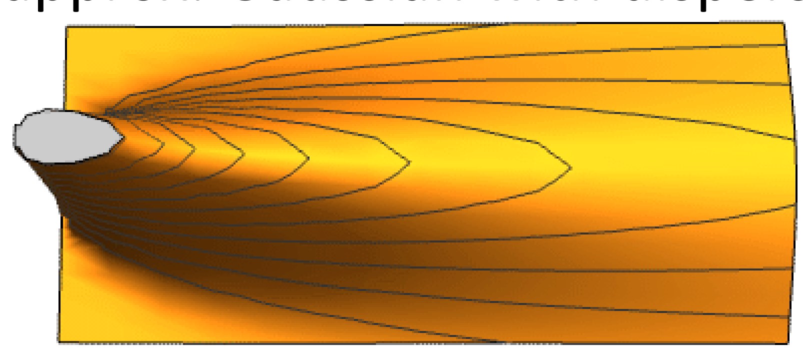
We assume that in the above momentum variable is ‘fast’ , described by angular diffusion in momentum space , and position is ‘slow’, so that

$$p = u(\cos\theta_p, \sin\theta_p)$$

$$f = f_q \cdot f_p, \quad f_p = \frac{e^{\frac{-\theta_p{}^2}{4\,\kappa\,t}}}{\sqrt{4\pi\kappa t}}, \qquad f_q = \langle f_q^0(q - p\,t) \rangle = \int f_q^0(q - p\,t)\, f_p d\theta_p$$

Starting with Gaussian f^0_q it will stay approx. Gaussian with dispersion in transversal direction

$$\sigma_t = \sqrt{\sigma_0^2 + 4\,\kappa\,t\,u^2}$$



MZ

Decision making

Certainty

Risk

Uncertainty