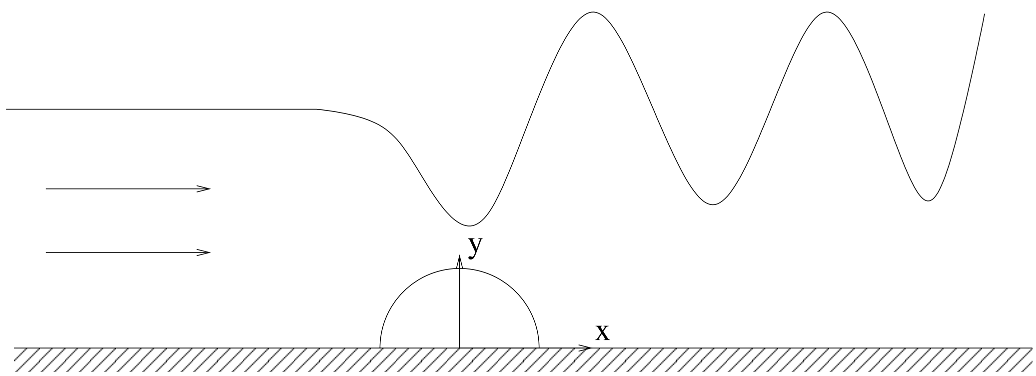
NUMERICAL INVESTIGATION OF THE PROPERTIES OF NONLINEAR FREE SURFACE FLOWS

Jean-Marc Vanden-Broeck
University College London

Cambridge, March 2017


y
x

J. Fluid Mech. (1997), *vol.* 342, *pp.* 403–409. Printed in the United Kingdom

An axisymmetric free surface with a 120 degree angle along a circle

By J.-M. VANDEN-BROECK[1] AND JOSEPH B. KELLER[2]

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(Received 24 December 1996 and in revised form 3 March 1997)

Singularities on Free Surfaces of Fluid Flows

By P. Milewski, J.-M. Vanden-Broeck, and Joseph B. Keller

Isolated singularities on free surfaces of two-dimensional and axially symmetric three-dimensional steady potential flows with gravity are considered. A systematic study is presented, where known solutions are recovered and new ones found. In two dimensions, the singularities found include those described by the Stokes solution with a 120° angle, Craya's flow with a cusp on the free surface, Gurevich's flow with a free surface meeting a rigid plane at 120° angle, and Dagan and Tulin's flow with a horizontal free surface meeting a rigid wall at an angle less than 120°. In three dimensions, the singularities found include those in Garabedian's axially symmetric flow about a conical surface with an approximately 130° angle, flows with axially symmetric cusps, and flows with a horizontal free surface and conical stream surfaces. The Stokes, Gurevich, and Garabedian flows are exact solutions. These are used to generate local solutions, including perturbations of the Stokes solution by Grant and Longuet-Higgins and Fox, perturbations of Gurevich's flow by Vanden-Broeck and Tuck, asymmetric perturbations of Stokes flow and nonaxisymmetric perturbations of Garabedian's flow. A generalization of the Stokes solution to three fluids meeting at a point is also found.

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STUDIES IN APPLIED MATHEMATICS 100:245–267 245

Published by Blackwell Publishers, 350 Main Street, Malden, MA 02148, USA, and 108 Cowley Road, Oxford, OX4 1JF, UK.

Rising bubbles in a two-dimensional tube with surface tension, Phys. Fluids, 27 (1984), pp. 2604-2607.

Alex Doak and VdB (2017)

SINGULARITIES....

two-dimensional flows

axisymmetric flows

three-dimensional flows

TWO-DIMENSIONAL FLOWS

fluid is inviscid and incompressible

flow is irrotational

steady

FORMULATION

$$u = \phi_x = \psi_y \quad \text{and} \quad v = \phi_y = -\psi_x$$

$$\phi_{xx} + \phi_{yy} = 0$$

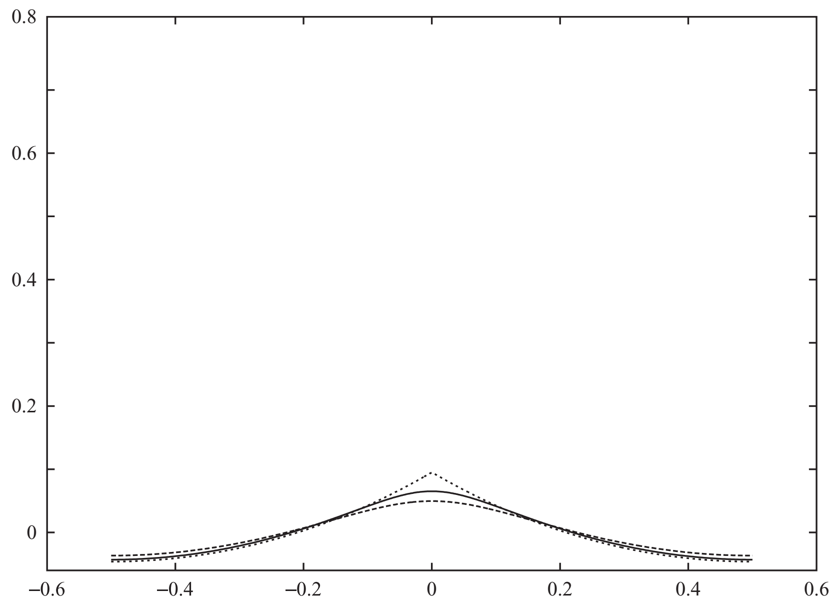
$$\phi_y = \phi_x \zeta_x \quad \text{on} \quad y = \zeta(x)$$

$$\frac{1}{2}(\phi_x^2 + \phi_y^2) + gy - \frac{T}{\rho}\frac{\zeta_{xx}}{(1+\zeta_x^2)^{1/3}} = B \quad \text{on} \quad y = \zeta(x)$$

$$\phi_n = 0 \quad \text{on} \quad \text{rigid boundaries}$$

$f = \phi + i\psi$: analytic functions of $z = x + iy$

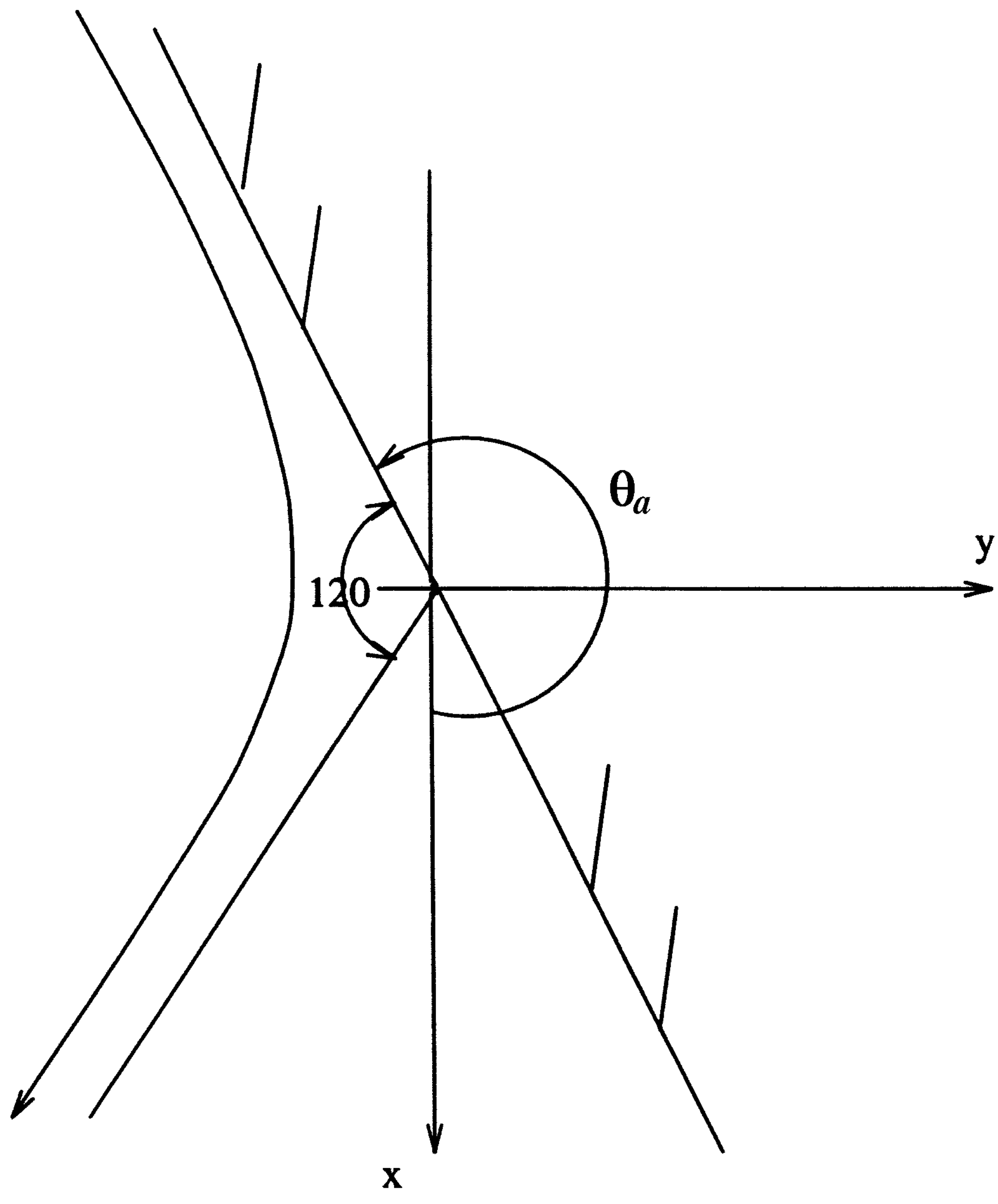
$u - iv$: analytic function of z and of $f = \phi + i\psi$

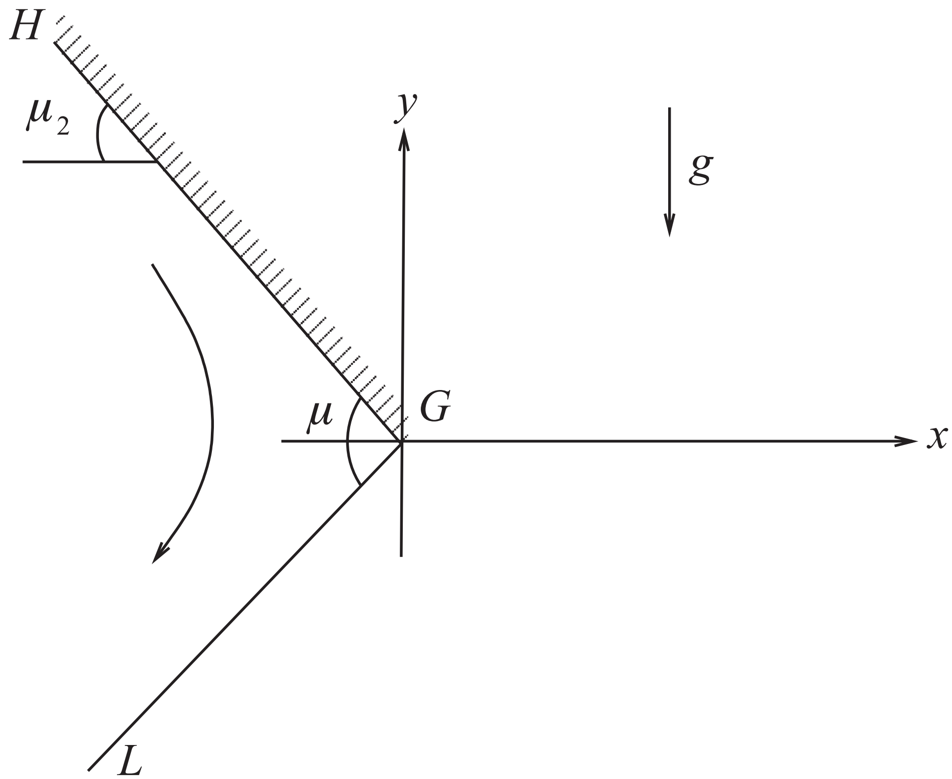


Stokes (1847), Michell (1883), Amick, Fraenkel and Toland (1982)

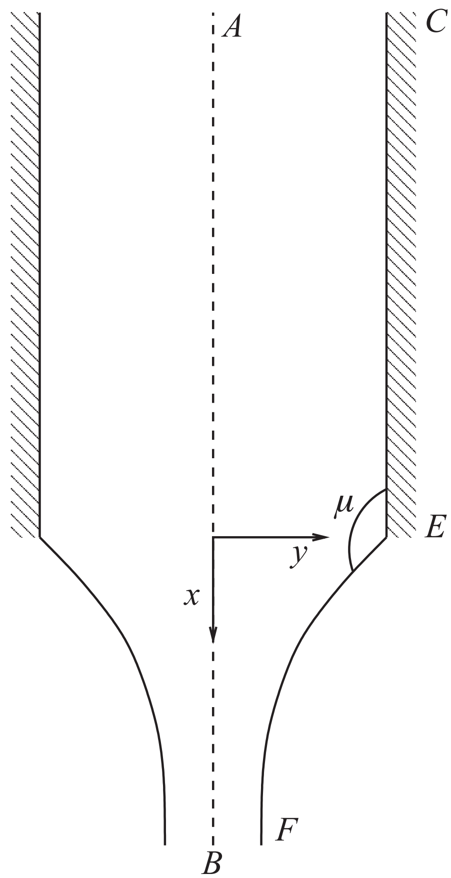
Olfe and Rottman (1980)

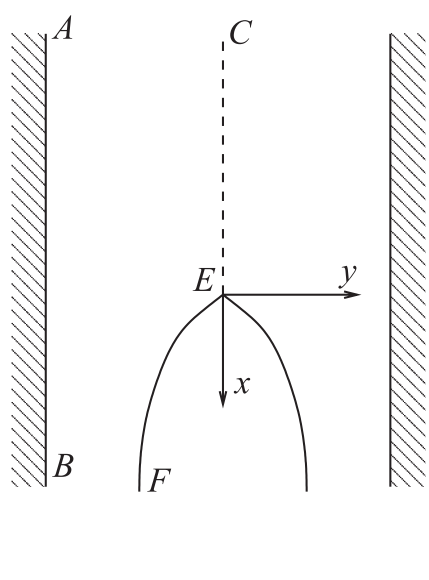
Vdb (2012)


θ_a
y
120
x



$$\mu = \frac{2\pi}{3}, \quad \mu = \frac{\pi}{2}, \quad \mu = \pi$$


A
C
μ
E
y
x
F
B


A
C
E
y
x
B
F

$$F = \frac{U}{(gH)^{1/2}}$$

$$\mu = \pi, \qquad \mu = \frac{2\pi}{3} \qquad \text{and} \qquad \mu = \frac{\pi}{2}$$

$$F_c \approx 0.36$$

$$\mu = \pi \quad \text{for} \quad F > F_c$$

$$\mu = \frac{\pi}{2} \quad \text{for} \quad F < F_c$$

$$\mu = \frac{2\pi}{3} \quad \text{for} \quad F = F_c$$

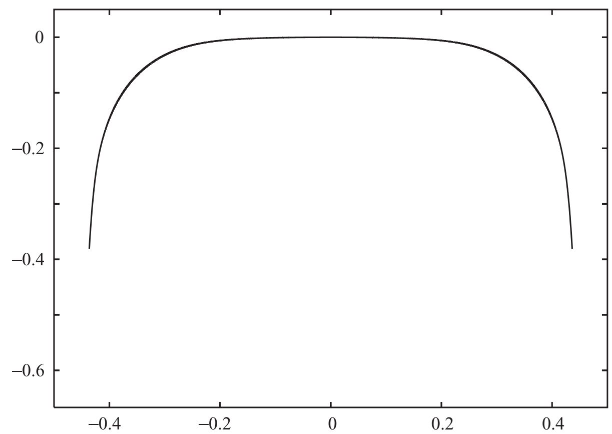


Fig. 3.48. Rising bubble in a tube for $F = 0.1$.

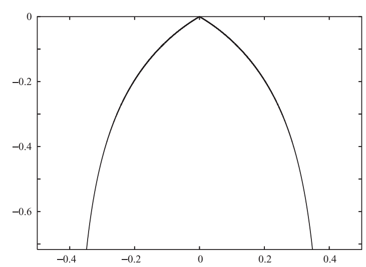


Fig. 3.50. Rising bubble in a tube for $F = F_c$. There is a 120° angle at the apex of the bubble.

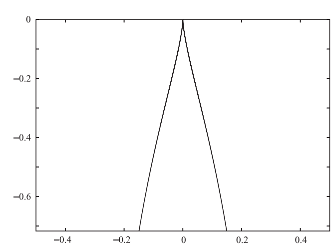
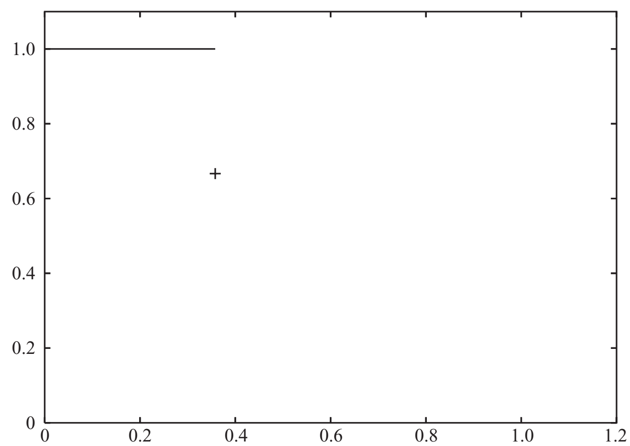
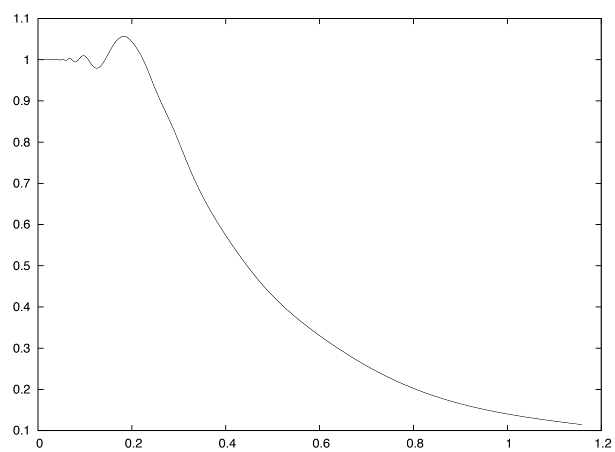
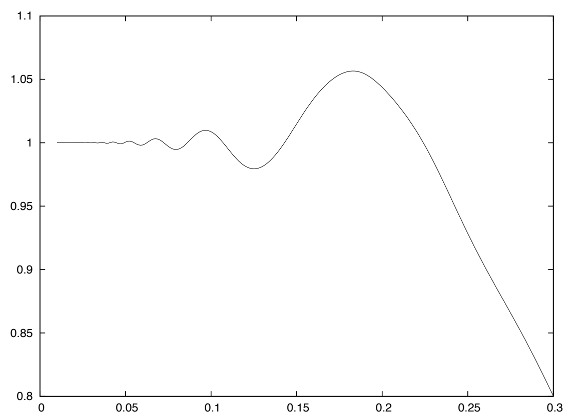


Fig. 3.52. Rising bubble in a tube for $F = 1$. There is a cusp at the apex of the free surface profile.

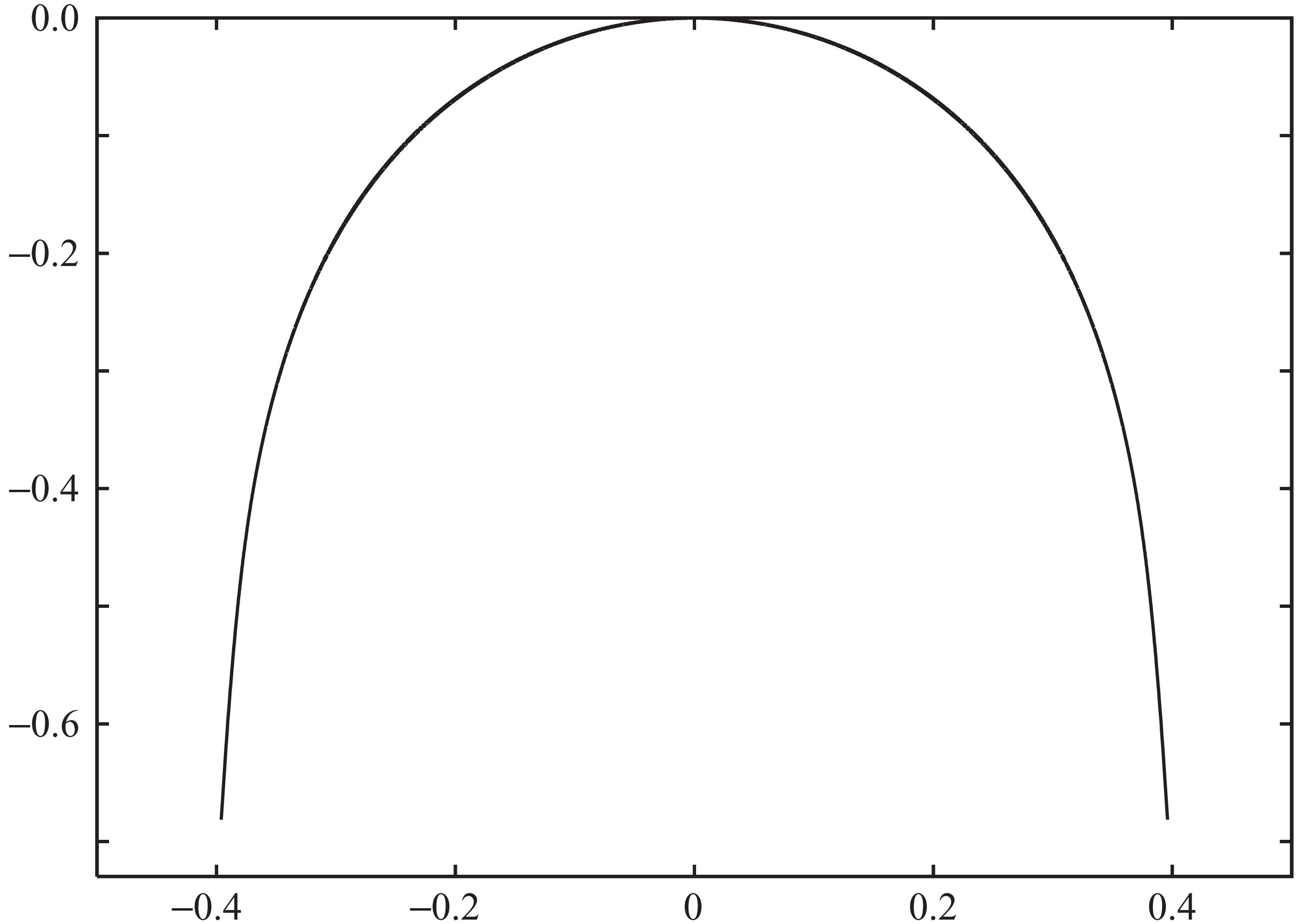
$$\nu = 2\frac{\pi - \mu}{\pi}$$







1.1
1.05
1
0.95
0.9
0.85
0.8
0
0.05
0.1
0.15
0.2
0.25
0.3



F experimental 0.25

F numerical 0.23

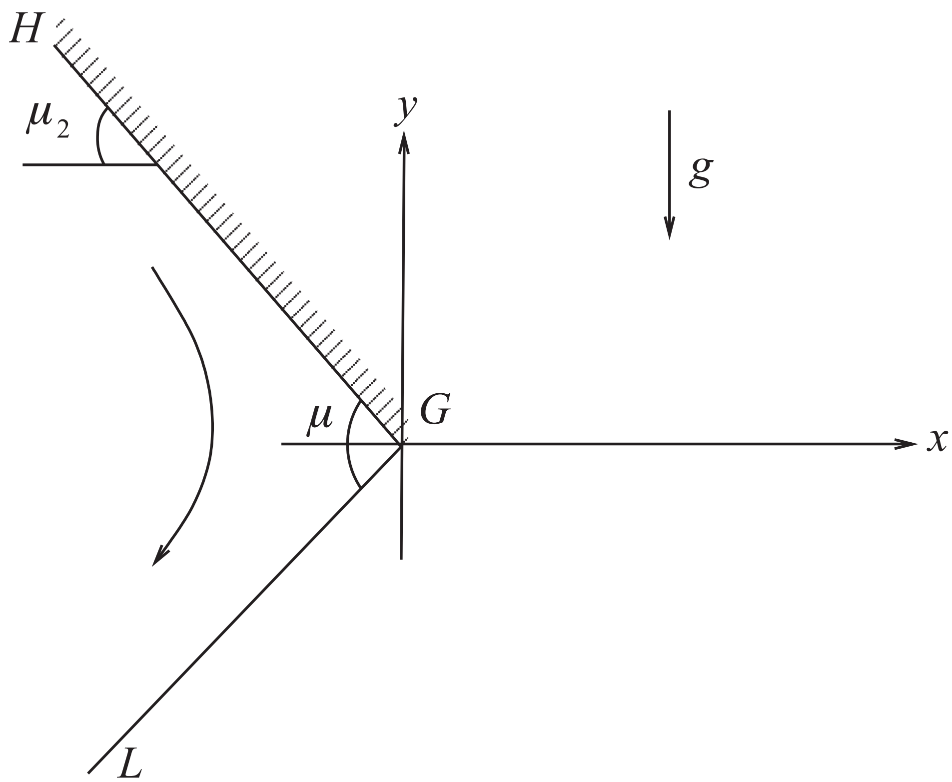
Fingers in a Hele Shaw cell

Vdb, 1983, Fingers in a Hele-Shaw cell with surface tension. Phys. Fluids 26, 2033 –2034.

AXISYMMETRIC FLOWS

Alex Doak...

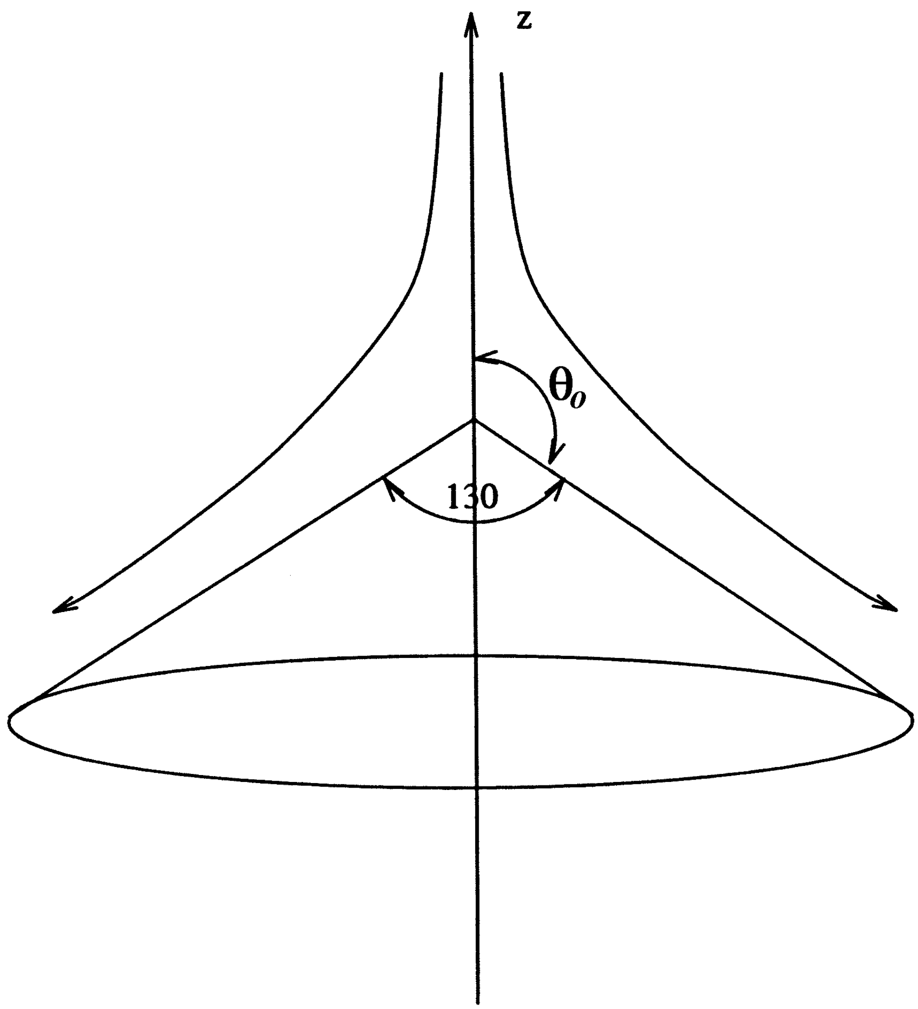
Recall the two-dimensional case ....

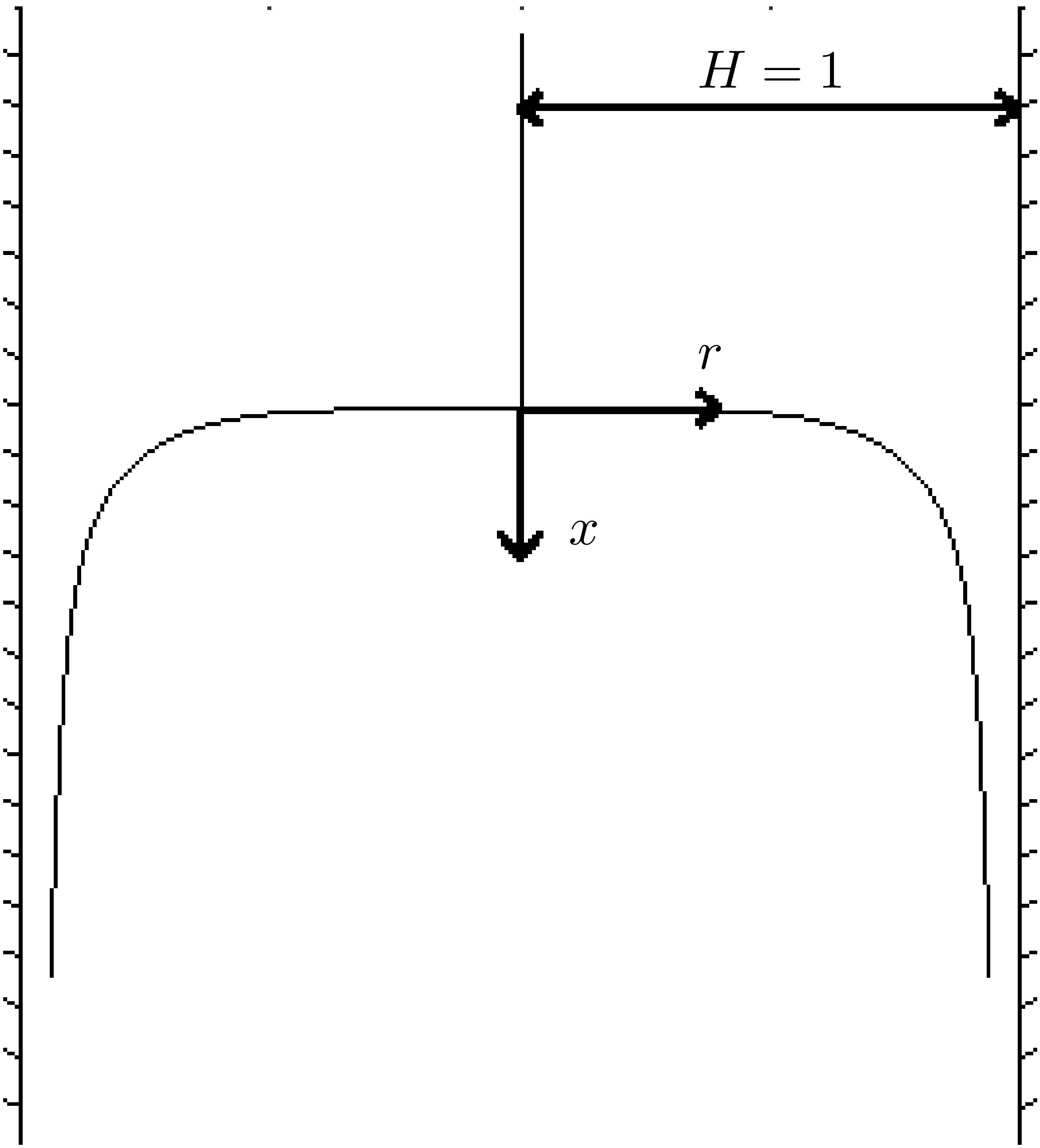


$$\mu = 120^0, \quad \mu = 90^0, \quad \mu = 180^0$$

Now take $\mu_2 = \frac{\pi}{2}$ and rotate.....then

$$\mu = 115^0, \quad \mu = 90^0, \quad \mu = 180^0$$

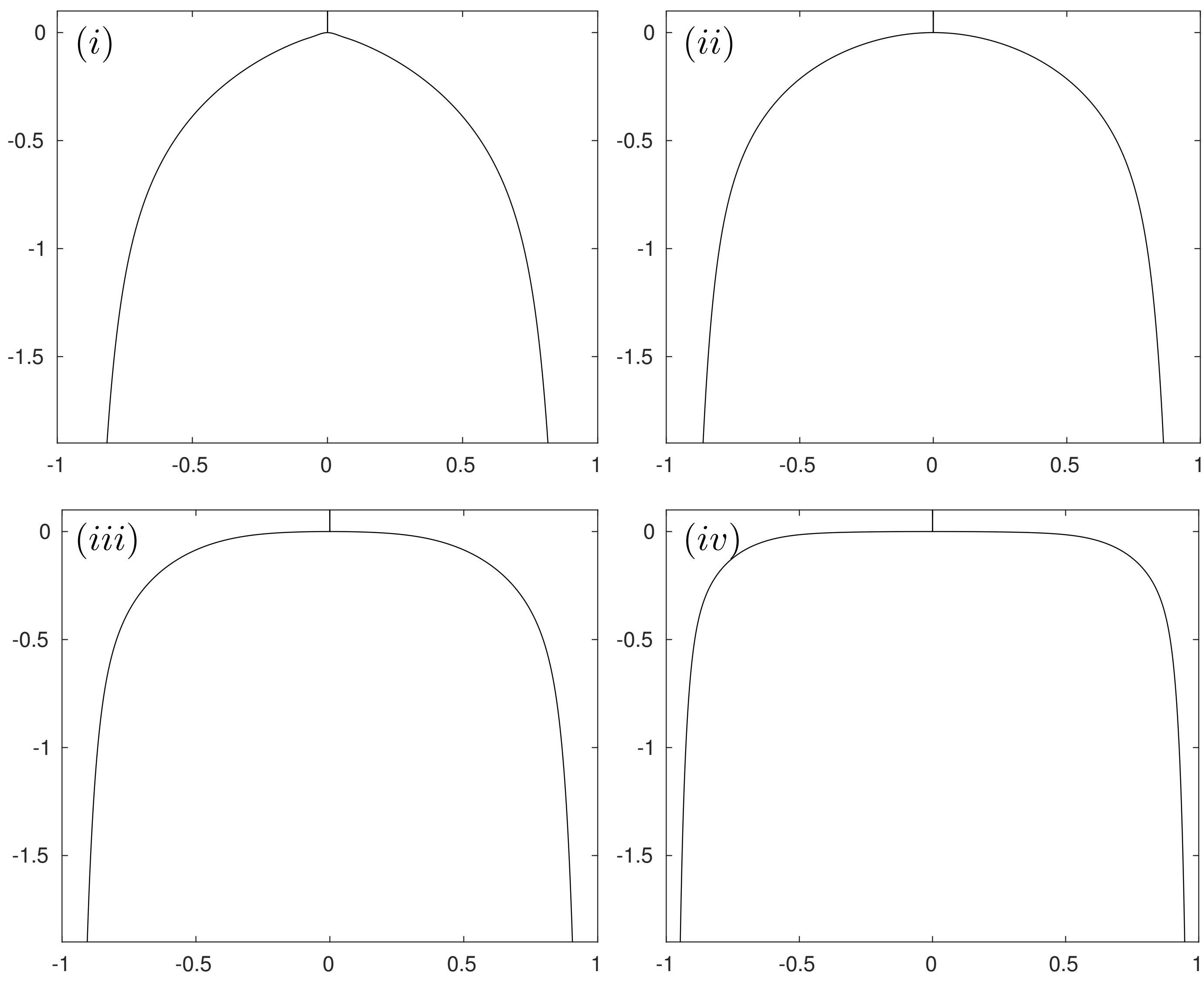

z
θ_o
130


$H = 1$
r
x

R.M. Davies and G.I. Taylor (1950)

Previous work: H. Levine and Y. Yang (1989)

L.C. Woods (1951, 1953)



(i)
(ii)
(iii)
(iv)
0
-0.5
-1
-1.5
-1
-0.5
0
0.5
1

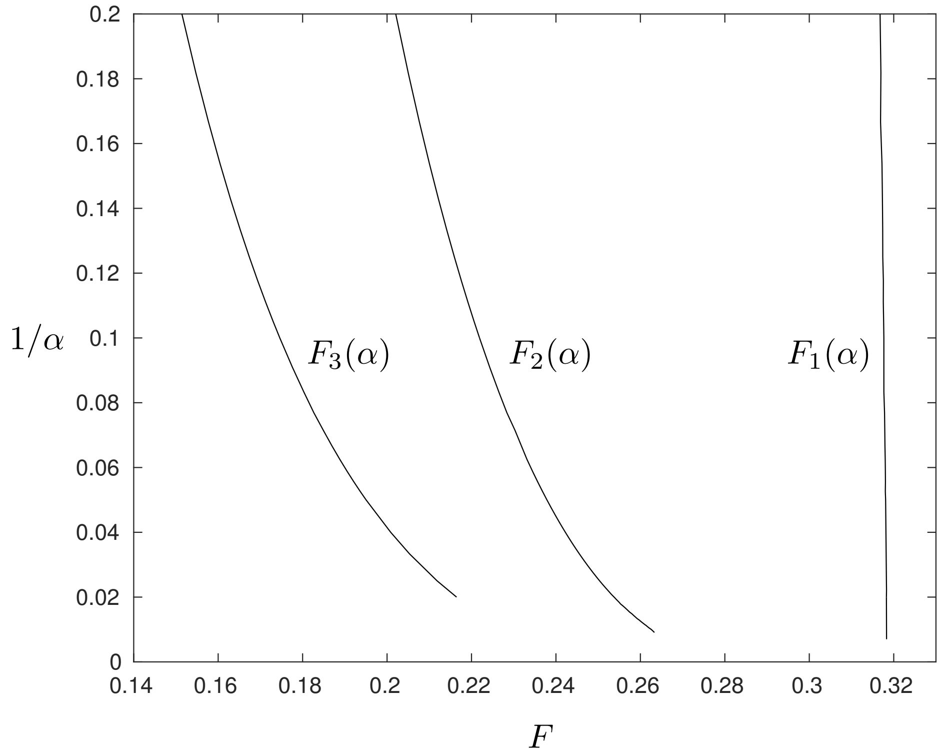


Figure 6: The first three $\mu = \pi/2$ solution branches F_1, F_2 and F_3 for the 2D problem. In the limit $\alpha \to \infty$, the solution branches approach $F^* \approx 0.318$

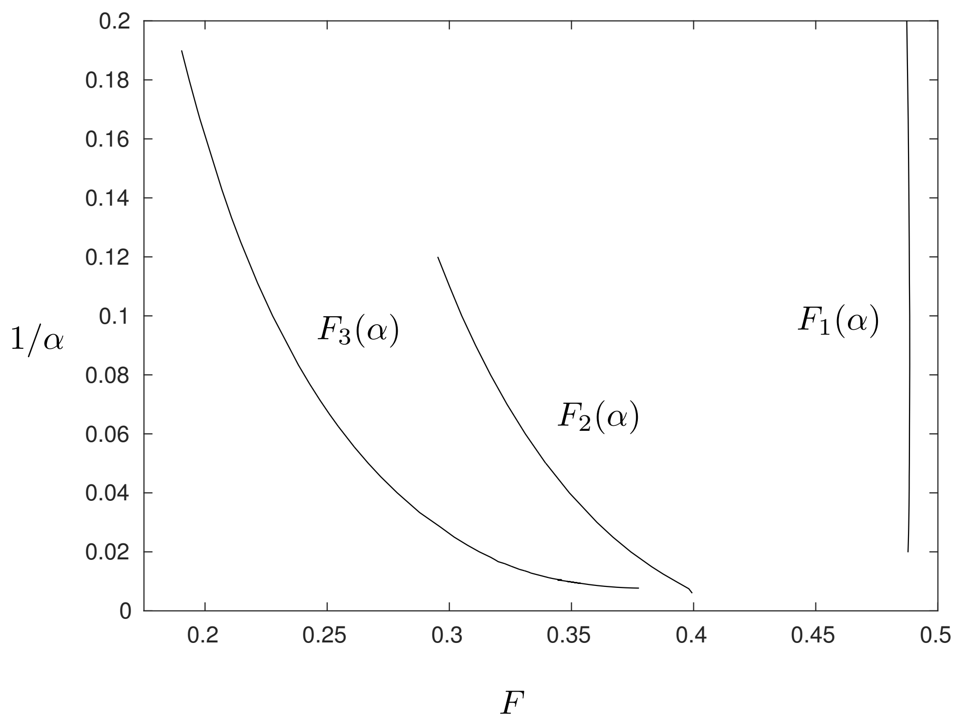


Figure 7: The first three solution branches F_1, F_2 and F_3 for the 3D problem. In the limit $\alpha \to \infty$, the solution branches approach $F^* \approx 0.49$

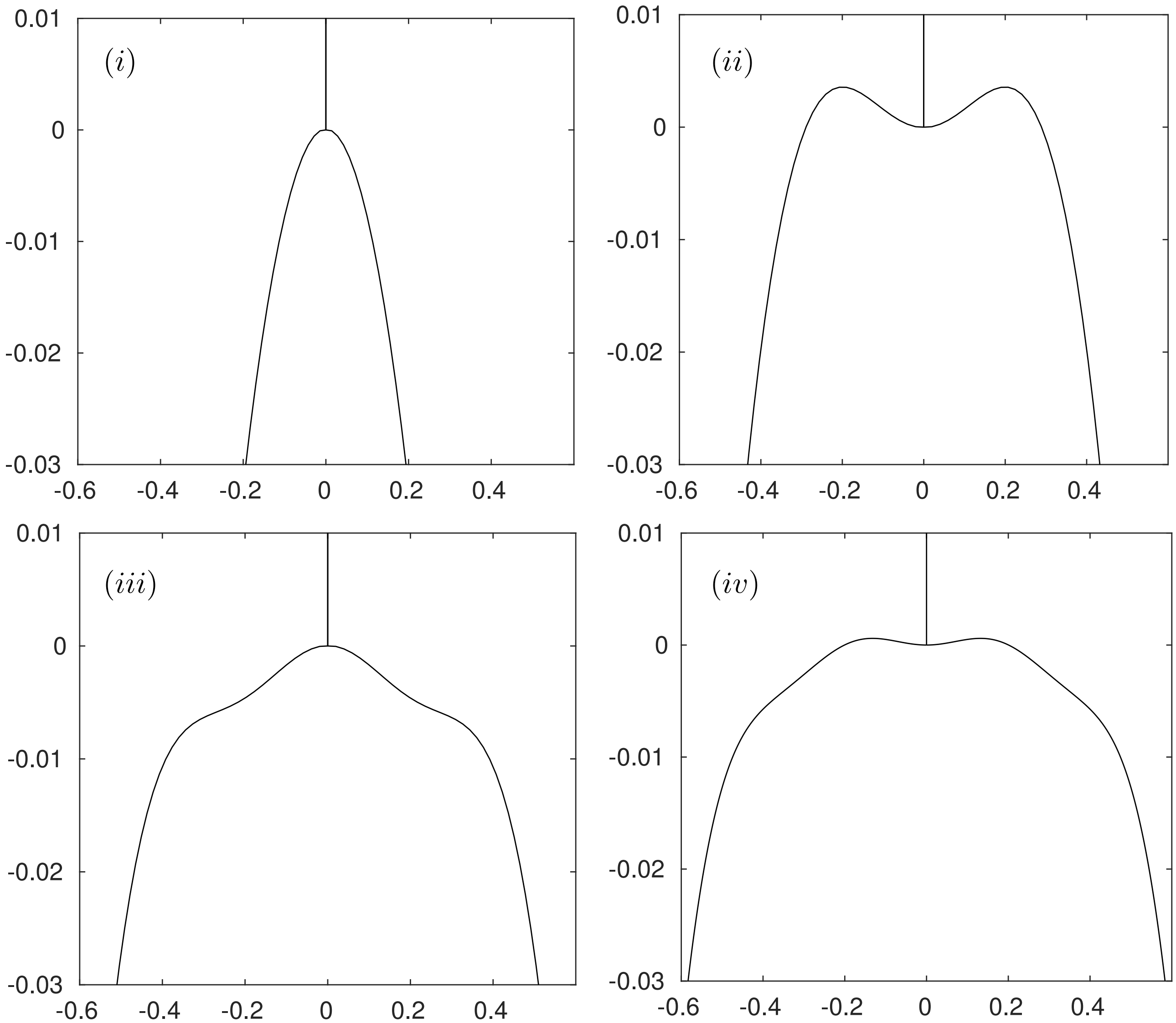


Figure 8: Solutions to the 2D problem with $\mu = \pi/2$ from the first four branches. Every solution is given by $\alpha = 5$, and the values of F are (i) $F_1(5) = 0.316$, (ii) $F_2(5) = 0.202$, (iii) $F_3(5) = 0.151$ and (iv) $F_4(5) = 0.122$.

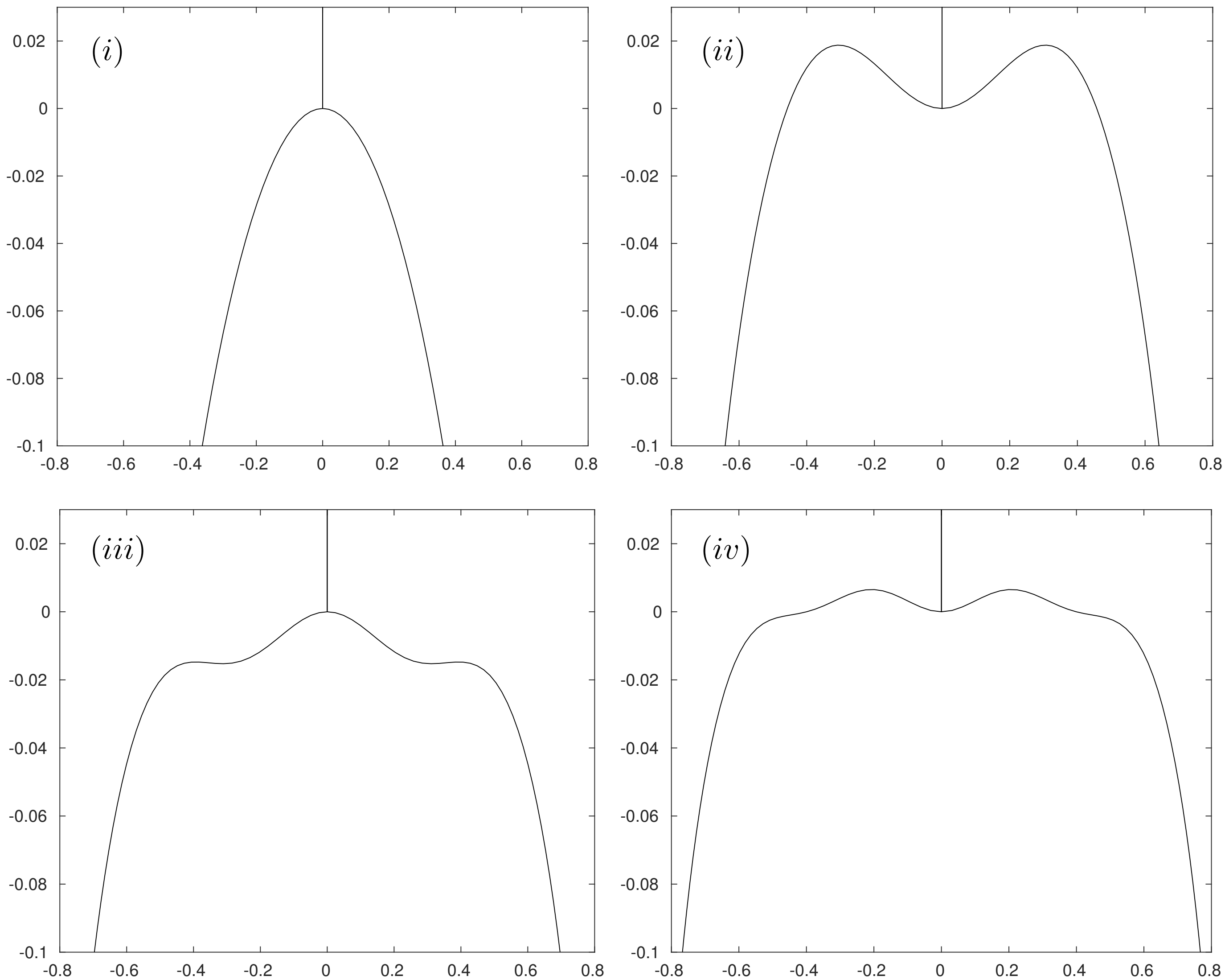
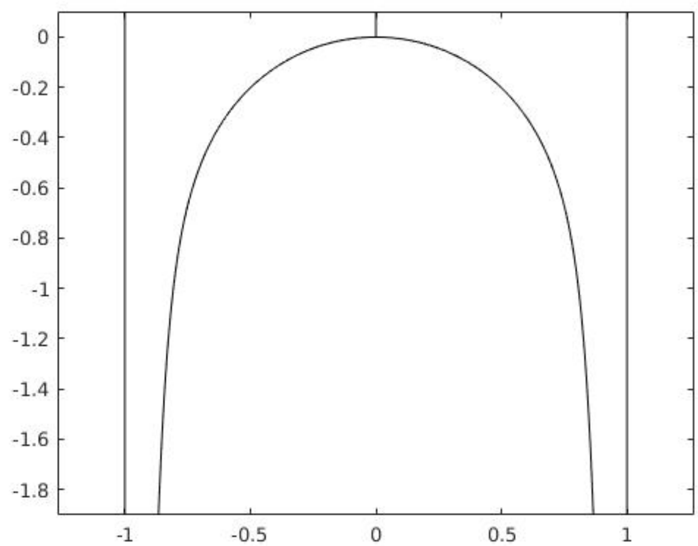
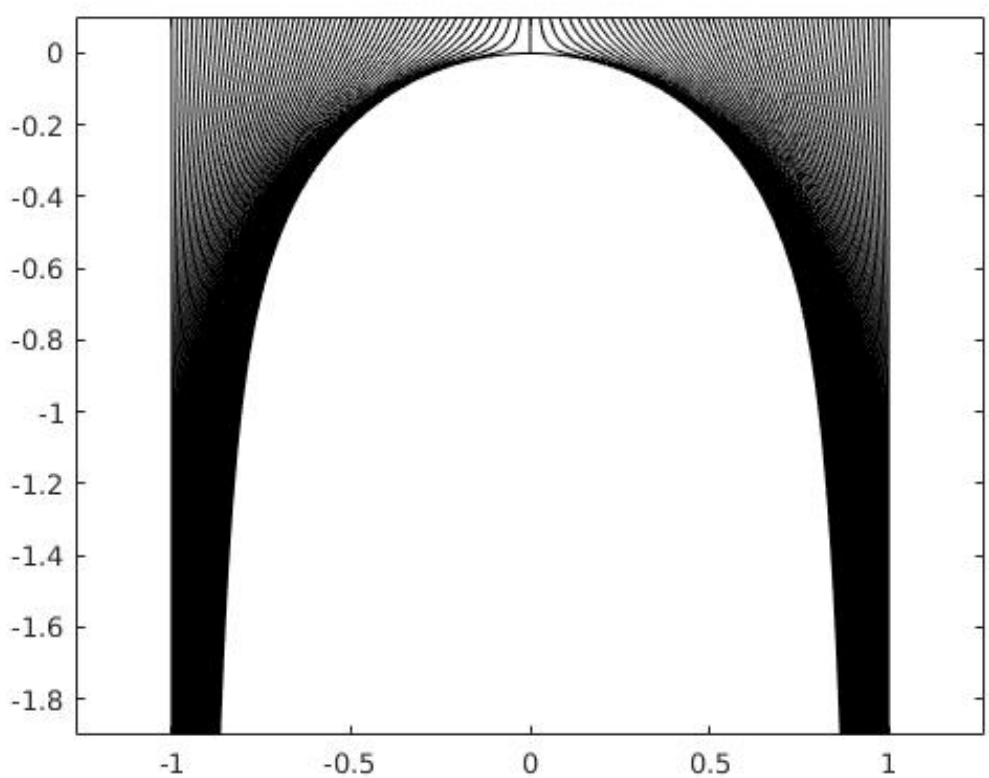


Figure 9: Solutions to the 3D problem with $\mu = \pi/2$ from the first three branches. Every solution is given by $\alpha = 10$, and the values of F are (i) $F_1(10) = 0.488$, (ii) $F_2(10) = 0.305$ and (iii) $F_3(10) = 0.228$



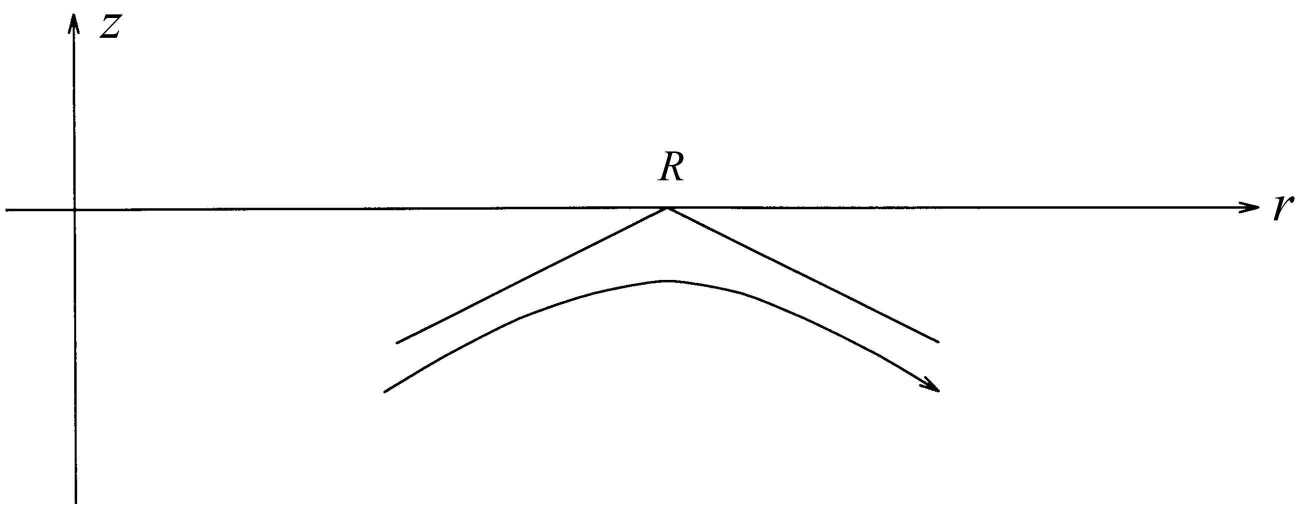
F experimental 0.48

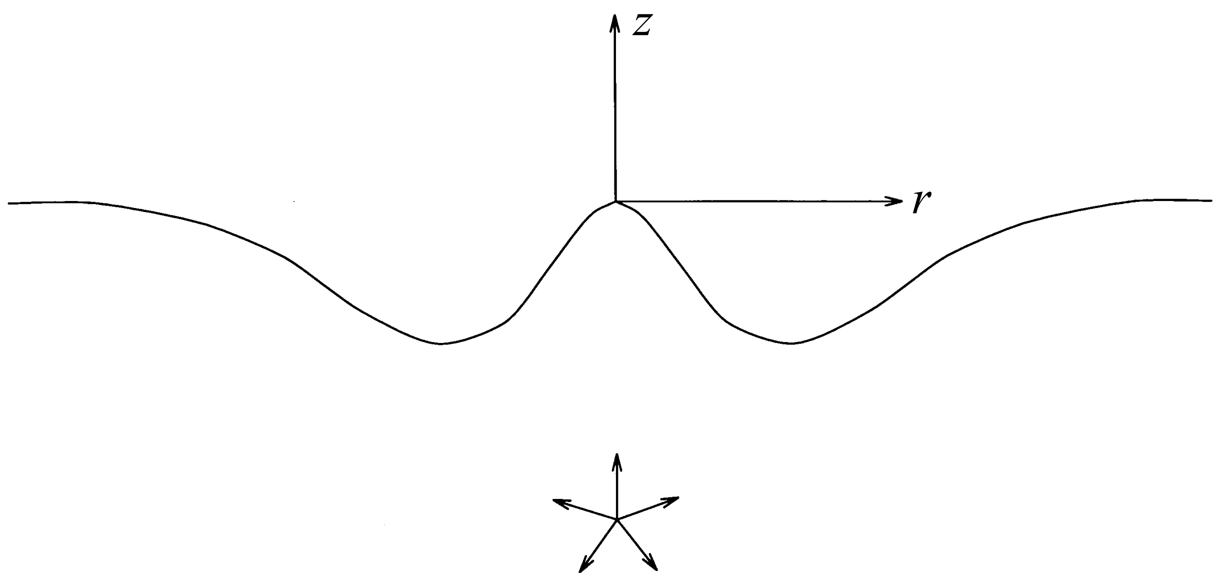
F numerical 0.49


0
-0.2
-0.4
-0.6
-0.8
-1
-1.2
-1.4
-1.6
-1.8
-1
-0.5
0
0.5
1

THREE DIMENSIONAL FLOWS

Keller and VdB (1997)


z
R
r


z
r

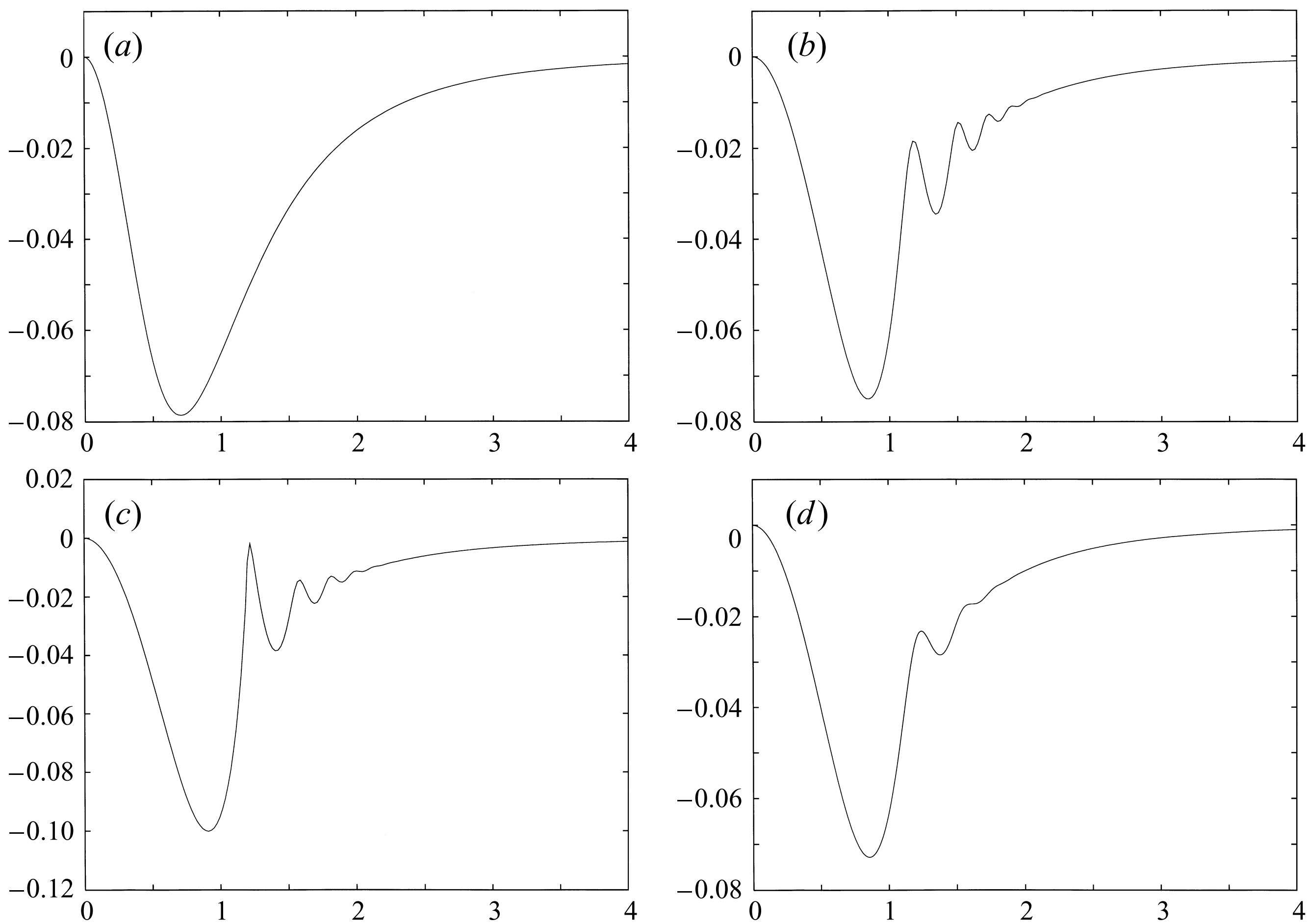
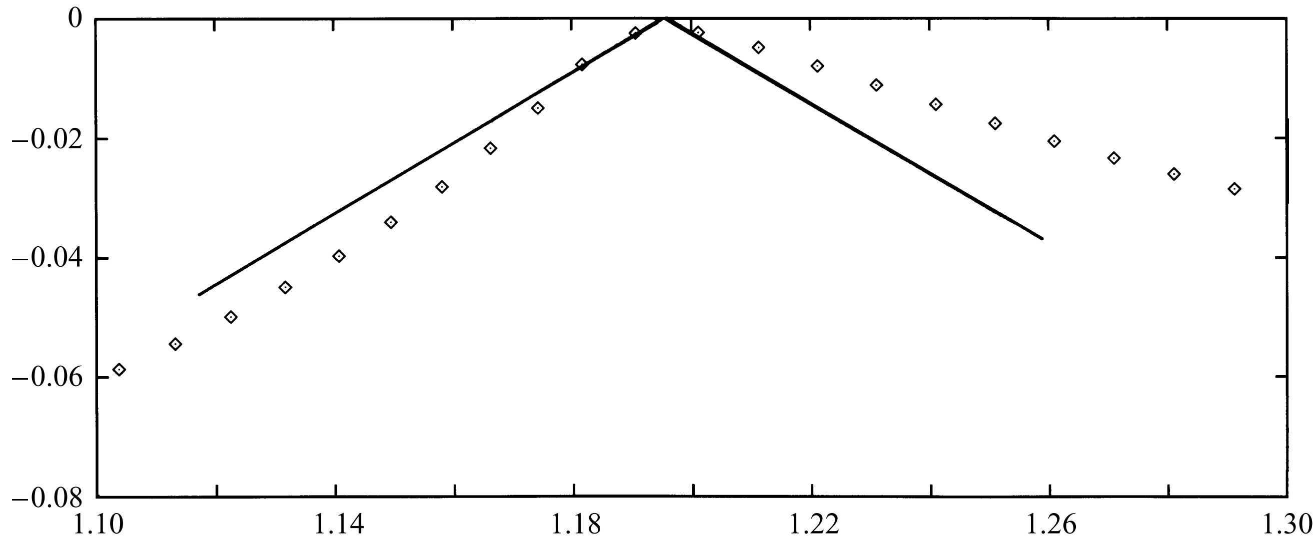


FIGURE 2. Free-surface profiles in the (r, z)-plane for (a) $F = 2$, (b) $F = 5$, (c) $F = F_c \approx 5.4$, (d) $F = 5$. The profile (d) differs from the one in (b) because the truncation value of r is not large enough.


0
−0.02
−0.04
−0.06
−0.08
1.10
1.14
1.18
1.22
1.26
1.30

CONCLUSIONS

two-dimensional free surface flows

axisymmetric free surface flows

three-dimensional free surface flows