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The Unified Transform, Imaging and Asymptotics

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I. The Unified Transform

$$u_t + u_x + u_{xxx} = 0, \quad 0 < x < \infty, \quad t > 0,$$
$$u(x,0) = u_0(x), \quad 0 < x < \infty,$$
$$u(0,t) = g_0(t), \quad t > 0.$$

Example 1: The Heat Equation

$$u_t = u_{xx}, \quad 0 < x < \infty, \ \ 0 < t < T, \quad T > 0,$$
$$u(x,0) = u_0(x), \ \ 0 < x < \infty, \quad u(0,t) = g_0(t), \ \ 0 < t < T.$$

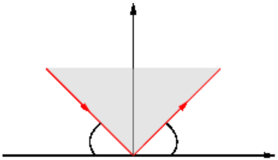
The classical sine transform yields

$$u(x,t) = \frac{2}{\pi} \int_0^\infty e^{-k^2 t} \sin(kx) \Big[\int_0^\infty \sin(k\xi) u_0(\xi) d\xi - k \int_0^t e^{k^2 s} g_0(s) ds \Big] dk.$$

The unified transform yields

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-k^2 t}\hat{u}_0(k)dk - \frac{1}{2\pi}\int_{\partial D^+} e^{ikx-k^2 t}\left[\hat{u}_0(-k) + 2ikG_0(k^2)\right] dk,$$

$$\hat{u}_0(k) = \int_0^{\infty} e^{-ikx}u_0(x)dx, \quad \mathrm{Im}k \le 0, \quad G_0(k) = \int_0^T e^{ks}g_0(s)ds, \quad k \in \mathbb{C}.$$



Alternatively, $G_0(k,t) = \int_0^t e^{ks}g_0(s)ds, \quad k \in \mathbb{C}.$

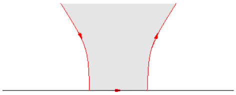
Example 2: The Stokes Equation

The unified transform yields

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-(ik-ik^3)t}\hat{u}_0(k)dk - \frac{1}{2\pi}\int_{\partial D^+} e^{ikx-(ik-ik^3)t}\tilde{g}(k)dk,$$

$$\tilde{g}(k) = \frac{1}{\nu_1 - \nu_2}\left[(\nu_1 - k)\hat{u}_0(\nu_2) + (k-\nu_2)\hat{u}_0(\nu_1)\right] + (3k^2-1)G_0(\omega(k)),$$

$$\omega(k) = ik - ik^3, \quad k \in D^+, \quad D^+ = \left\{\mathrm{Re}\omega(k) < 0\right\} \cap \mathbb{C}^+.$$



$$\nu_1, \nu_2 : \quad \omega(k) = \omega(\nu k).$$

$$\nu_j^2 + k\nu_j + k^2 - 1 = 0, \quad j = 1, 2.$$

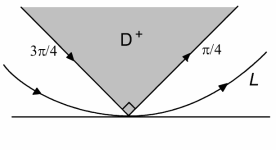
Numerical Implementation

Consider the heat equation with

$$u_0(x) = x\exp(-a^2 x), \quad g_0(t) = \sin bt, \qquad a, b > 0.$$

Then,

$$u(x,t) = \frac{1}{2\pi}\int_L e^{ikx-k^2 t}\left[\frac{1}{(ik+a)^2} - \frac{1}{(-ik+a)^2} - k\left(\frac{e^{(k+ib)t}-1}{k+ib} - \frac{e^{(k-ib)t}-1}{k-ib}\right)\right] dk.$$



- N. Flyer and A.S. Fokas, A hybrid analytical numerical method for solving evolution partial differential equations. I. The half-line, Proc. R. Soc. 464, 1823-1849 (2008)

References

- B. Fornberg and N. Flyer, A numerical implementation of Fokas boundary integral approach: Laplace's equation on a polygonal domain, Proc. R. Soc. A 467, 2983–3003 (2011)
- A.C.L. Ashton, On the rigorous foundations of the Fokas method for linear elliptic partial differential equations, Proc. R. Soc. A 468, 1325–1331 (2012)
- P. Hashemzadeh, A.S Fokas and S.A. Smitheman, A numerical technique for linear elliptic partial differential equations in polygonal domains, Proc. R. Soc. A 471, 20140747 (2015)
- M.J. Ablowitz, A.S. Fokas and Z.H. Musslimani, On a new nonlocal formulation of water waves, J. Fluid Mech. 562, 313–344 (2006)
- D. Crowdy, Geometric function theory: a modern view of a classical subject, Nonlinearity 21, 205–219 (2008)
- N.E. Sheils and D.A. Smith, Heat equation on a network using the Fokas method, J. Phys. A 48(33), 335001 (2015)
- D.M. Ambrose and D.P. Nicholls, Fokas integral equation for three dimensional layered-media scattering, J. Comput. Phys. 276, 1–25 (2014)

Summary

1747, d' Alembert and Euler: Separation of Variables
1807, Fourier: Transforms
1814, Cauchy: Analyticity
1828, Green: Green's Representations
1845, Kelvin: Images

Fourier (spectral) Space

Transforms

Physical Space

Green's Integral Representations
Method of Images

New Method

Integral Representations in the Fourier Space
Invariance of Global Relation and Jordan's Lemma

http://www.personal.reading.ac.uk/ smr07das/UTM/

- A.S. Fokas, *A Unified Approach to Boundary Value Problems*, CBMS-NSF, SIAM (2008)
- A.S. Fokas and E.A. Spence, Synthesis as Opposed to Separation of Variables, SIAM Review 54, 291–324 (2012)
- B. Deconinck, T. Trogdon and V. Vasan, The Method of Fokas for Solving Linear Partial Differential Equations, SIAM Review 56, 159–186 (2014)

PET - SPECT

F-Gelfand (1994) : Novel derivation of 2D FT via $\bar{\partial}$

$$(\partial_{x_1} + i\partial_{x_2} - k)\,\mu(x_1, x_2, k) = u(x_1, x_2).$$

F-Novikov (1992) : Novel derivation of Radon transform

$$\left[\frac{1}{2}\left(k + \frac{1}{k}\right)\partial_{x_1} + \frac{1}{2\mathrm{i}}\left(k - \frac{1}{k}\right)\partial_{x_2}\right]\mu(x_1, x_2, k) = f(x_1, x_2).$$

Novikov (2002) : Derivation of attenuated Radon transform

$$\left[\frac{1}{2}\left(k+\frac{1}{k}\right)\partial_{x_1} + \frac{1}{2\mathrm{i}}\left(k-\frac{1}{k}\right)\partial_{x_2}\right]\mu(x_1,x_2,k) + f(x_1,x_2)\mu(x_1,x_2,k) = g(x_1,x_2).$$

- G.A. Kastis, D. Kyriakopulou, A. Gaitanis, Y. Fernandez, B.F. Hutton and A.S. Fokas, Evaluation of the spline reconstruction technique for PET, Med. Phys. 41, 042501 (2014)
- G.A. Kastis, A. Gaitanis, A. Samartzis and A.S. Fokas, The SRT reconstruction algorithm for semiquantification in PET imaging, Med. Phys. 42, 5970–5982 (2015)

ATTENUATED RADON TRANSFORM

Direct Attenuated Radon transform

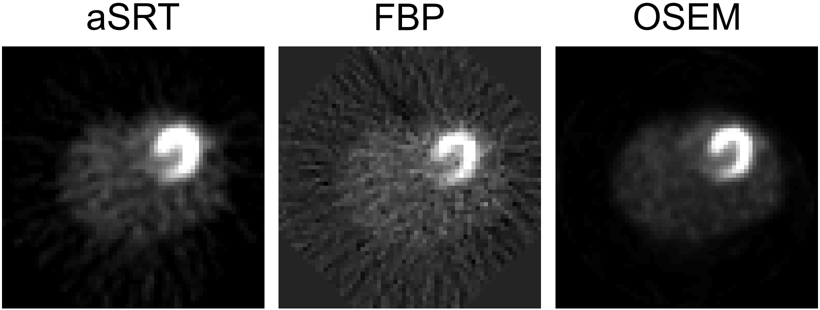
$$\hat{g}_f(\rho,\theta) = \int_{-\infty}^{\infty} e^{-\int_\tau^\infty f ds} g d\tau.$$

Inverse Attenuated Radon transform

$$P^{\pm} g(\rho) = \pm \frac{g(\rho)}{2} + \frac{1}{2i\pi} \oint_{-\infty}^{\infty} \frac{g(\rho')}{\rho' - \rho} d\rho'.$$

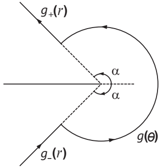
$$H(\theta,\tau,\rho) = e^{\int_\tau^\infty f ds} \left\{ e^{P^- \hat{f}(\rho,\theta)} P^- e^{-P^- \hat{f}(\rho,\theta)} + e^{-P^+ \hat{f}(\rho,\theta)} P^+ e^{P^+ \hat{f}(\rho,\theta)} \right\} \hat{g}_f(\rho,\theta).$$

$$g(x_1,x_2) = \frac{1}{4\pi}(\partial_{x_1} - i\partial_{x_2}) \int_0^{2\pi} e^{i\theta} H(\theta, x_1\cos\theta + x_2\sin\theta, x_2\cos\theta - x_1\sin\theta) d\theta.$$


aSRT
FBP
OSEM

III. Asymptotics of the Riemann Zeta Function

The Laplace equation in the exterior of the Hankel contour and the Riemann hypothesis



$$g_\pm(r) = \left(re^{\pm i\alpha}\right)^s u\left(re^{\pm i\alpha}\right), \;\; g(\theta) = -\frac{i}{a}\left(ae^{i\theta}\right)^s u\left(ae^{i\theta}\right), \;\; u(z) = \frac{1}{e^{-z}-1}.$$

The Riemann hypothesis is valid if and only if the above Neumann boundary value problem does not have a solution which is bounded as $r \to \infty$.

- A.S. Fokas and M.L. Glasser, The Laplace equation in the exterior of the Hankel contour and novel identities for the hypergeometric functions, Proc. R. Soc. A 469, 20130081 (2013)

On the Lindelöf Hypothesis

$$\frac{t}{\pi}\oint_{-\infty}^{\infty}\Re\left\{\frac{\Gamma(it-i\tau t)}{\Gamma(\sigma+it)}\Gamma(\sigma+i\tau t)\right\}|\zeta(\sigma+i\tau t)|^2\,\mathrm{d}\tau+\mathcal{G}(\sigma,t)=0,$$

$$0\le\sigma\le 1,\quad t>0,\quad (1)$$

$$\mathcal{G}(\sigma,t)=\begin{cases}\zeta(2\sigma)+\left(\frac{\Gamma(1-\bar{s})}{\Gamma(s)}+\frac{\Gamma(1-s)}{\Gamma(\bar{s})}\right)\Gamma(2\sigma-1)\zeta(2\sigma-1)+\frac{2(\sigma-1)\zeta(2\sigma-1)}{(\sigma-1)^2+t^2},\\ \Re\left\{\Psi\left(\frac{1}{2}+it\right)\right\}+2\gamma-\ln 2\pi+\frac{2}{1+4t^2},\end{cases}$$

with $\Psi(z)$ denoting the digamma function, i.e.,

$$\Psi(z)=\frac{\frac{\mathrm{d}}{\mathrm{dz}}\Gamma(z)}{\Gamma(z)},\quad z\in\mathbb{C},$$

and γ denoting the Euler constant.

We conclude that

$$\sum\sum_{m_1, m_2 \in N} \frac{1}{m_1^{s-it^{\delta_3}}} \frac{1}{m_2^{\bar{s}+it^{\delta_3}}} \frac{1}{\ln\left[\frac{m_2}{m_1}\left(t^{1-\delta_3}-1\right)\right]} = \begin{cases} O\left(t^{\delta_3}\ln t\right), & \sigma = \frac{1}{2}, \\ O\left(t^{\frac{\delta_3}{2}}\right), & \frac{1}{2} < \sigma < 1, \end{cases}$$

$$t \to \infty,$$

where N is defined by

$$N = \Big\{ m_1 \in \mathbb{N}^+, \quad m_2 \in \mathbb{N}^+, \quad 1 \le m_1 \le [T], \quad 1 \le m_2 < [T],$$

$$\frac{m_2}{m_1} > \frac{1}{t^{1-\delta_3}-1}\left(1 + c(t)\right), \; t^{-\frac{\delta_3}{2}} \prec c(t) \prec 1, \quad t > 0, \quad T = \frac{t}{2\pi} \Big\}.$$

References

- A. S. Fokas and J. Lenells, On the asymptotics to all orders of the Riemann zeta function and of a two-parameter generalization of the Riemann zeta function, https://arxiv.org/abs/1201.2633 (preprint)
- A. S. Fokas, On the proof of a variant of Lindelöf's hypothesis (preprint)