Joe Keller and the Courant Institute 1970-71: Beginning a Mathematical Journey

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Outline

- The Courant Institute 1970-71

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- The Watson Transformation in Obstacle Scattering

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- Nerve Impulse Propagation

Courant Institute

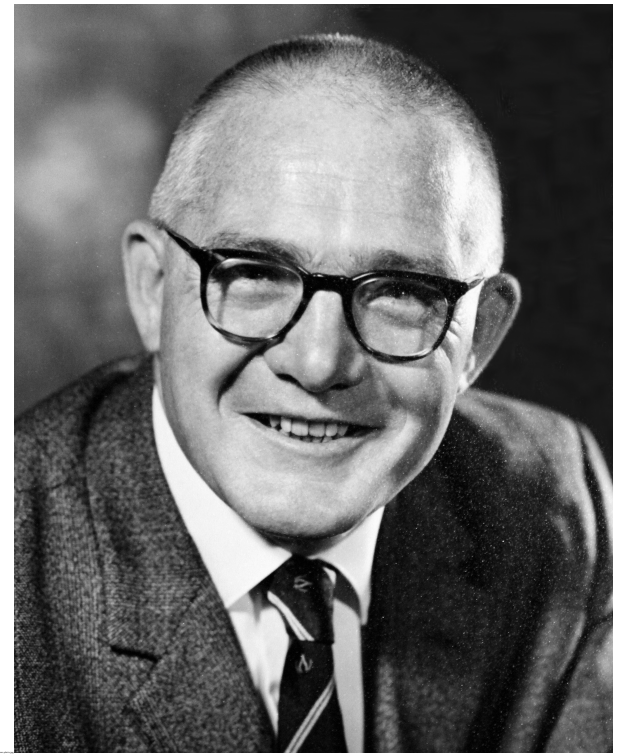

Figure : Douglas Jones

The story starts here!

Courant Institute

Figure : Courant Institute

Courant Institute

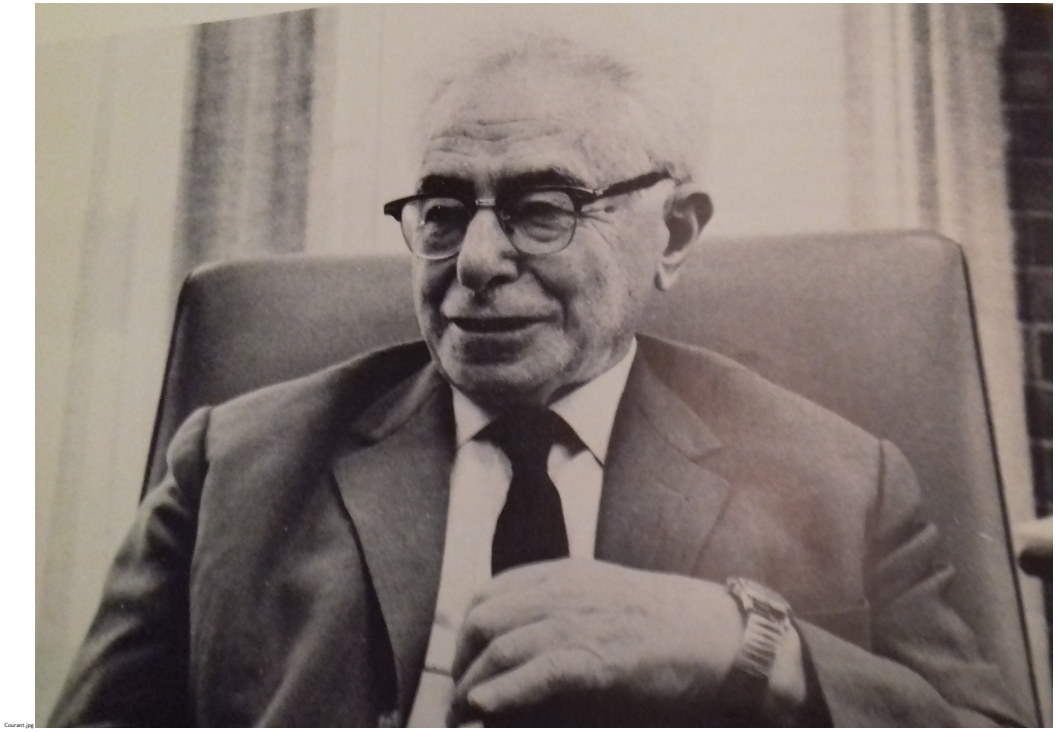

Figure : Richard Courant

Some Notable Faculty Jerome Berkowitz; Alexander Chorin; Richard Courant; Kurt Friedrichs; Paul Garabedian; James Glimm; Harold Grad;

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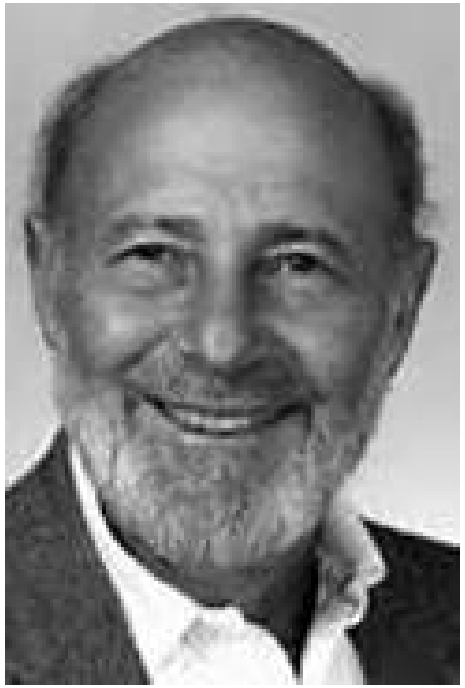

Figure : Joe

Warren Hirsch; Lars Hörmander; Eugene Isaacson; Fritz John; Joseph Keller:Morris Kline; Heinz-Otto Kreiss; Peter Lax: Donald Ludwig; Wilhelm Magnus; Henry McKean; Cathleen Morawetz;

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Jürgen Moser; Louis Nirenberg; Jacob Schwartz; James Stoker; Olof Widland; Shmuel Winograd.

Joe's Graduate Course

Special Topics in Applied Mathematics: Perturbation Theory and Asymptotics.

The Watson Transformation in Obstacle Scattering

G.N.Watson 1918, A.S.Fokas and E. A. Spence 2012, E.A. Spence 2014.

The acoustic scattering of an analytic incident field $u^i(r,\theta)$ by a sound soft circular cylinder of radius a is solved using separation of variables to give the scattered field $u(r,\theta)$ representation

$$u(r,\theta) = \frac{1}{2\pi} \sum_{n=-\infty}^{n=\infty} \frac{H_n^{(1)}(kr)}{H_n^{(1)}(ka)} e^{in\theta} U(a,-in), \tag{1}$$

where

$$U(a,-in) = -\int_0^{2\pi} e^{-in\omega} u^i(r,\omega) d\omega. \tag{2}$$

This series can be represented as the combined contour integrals

The Watson Transformation in Obstacle Scattering

$$u(r,\theta) = \frac{1}{2\pi}\left(\int_{-\infty+ic}^{\infty+ic} - \int_{-\infty-ic}^{\infty-ic}\right) \frac{e^{i\nu\theta}U(a,-i\nu)}{1-e^{-2\pi i\nu}} \frac{H_\nu^{(1)}(kr)}{H_\nu^{(1)}(ka)} d\nu. \tag{3}$$

Deform contour to surround simple the zeros of $H_\nu^{(1)}(ka)$. Rapid convergence.

The Watson Transformation in Obstacle Scattering

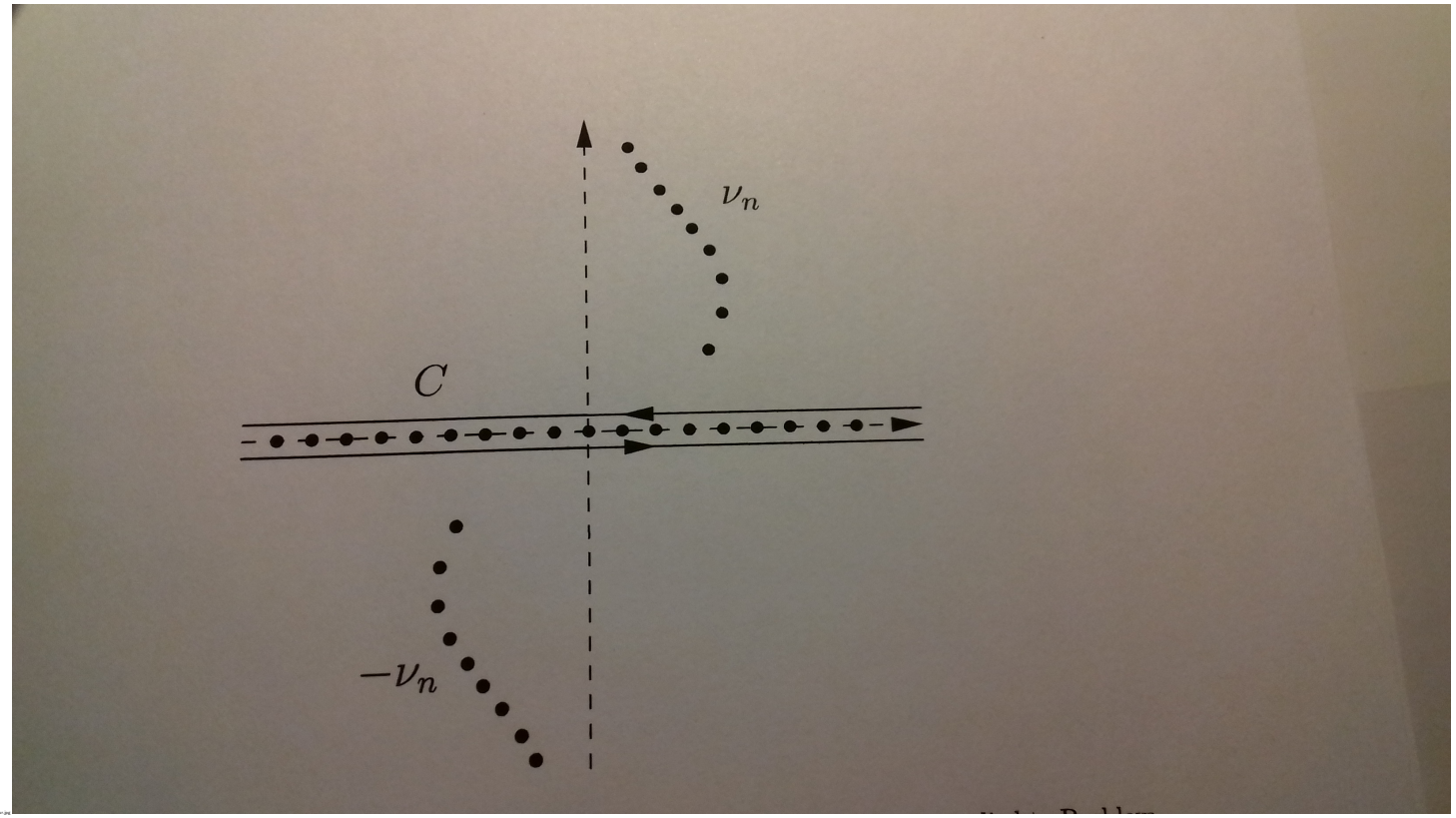


Figure : Poles and contours of the Watson Transformation, Spence 2014

The Watson Transformation in Obstacle Scattering

Obstacle scattering of a simple source by a prolate spheroid, BDS 1968, 1969.
Here we use Spheroidal coordinates

$$\begin{aligned} x &= c(\xi^2-1)^{1/2}(1-\eta^2)^{1/2}\cos\phi, \\ y &= c(\xi^2-1)^{1/2}(1-\eta^2)^{1/2}\sin\phi, \\ z &= c\xi\eta. \end{aligned}$$

$$\xi \in (1,\ \infty), \eta \in (-1,\ 1), \phi \in (0,\ 2\pi).$$

The Watson Transformation in Obstacle Scattering

Using separation of variables we encounter two separation constants $\mu,\ \nu$ in the spheroidal ODEs. Using the Watson idea we arrive at the double contour integral.

$$G(\xi,\ \eta,\ \phi;\ \xi^{'},\ \eta^{'},\ \phi^{'}) =$$

$$-\frac{1}{4\pi^2}\int_{\Gamma_\mu}\int_{\Gamma_\nu} G_1(\phi,\ \phi^{'};\ \mu)G_2(\eta,\ \eta^{'};\ \mu,\ \nu)G_3(\xi,\ \xi^{'};\ \mu,\ \nu)d\mu d\nu. \quad (4)$$

Here

$$G_1(\phi,\ \phi^{'};\ \mu) = -\frac{\cos(\mu(\pi - \mid \phi - \phi^{'} \mid)}{\sin(\mu\pi)}$$

and $G_2,\ G_3$ are complicated angular and radial Green's functions involving spheroidal wave functions.

The Watson Transformation in Obstacle Scattering

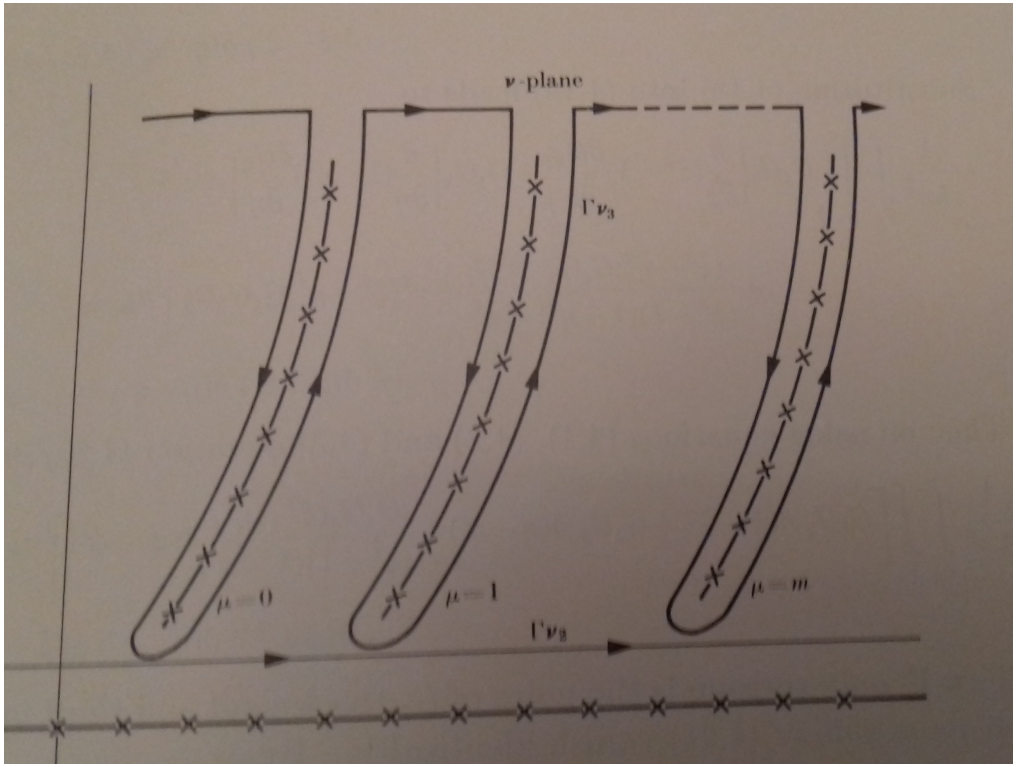


Figure : Double contours for the spheroid

Validity of the Geometrical Theory of Diffraction

Keller (1962), R. M. Lewis, N. Bleistein and D. Ludwig (1967), C. A. Bloom and B. J. Matkowsky (1969) BDS (1976).

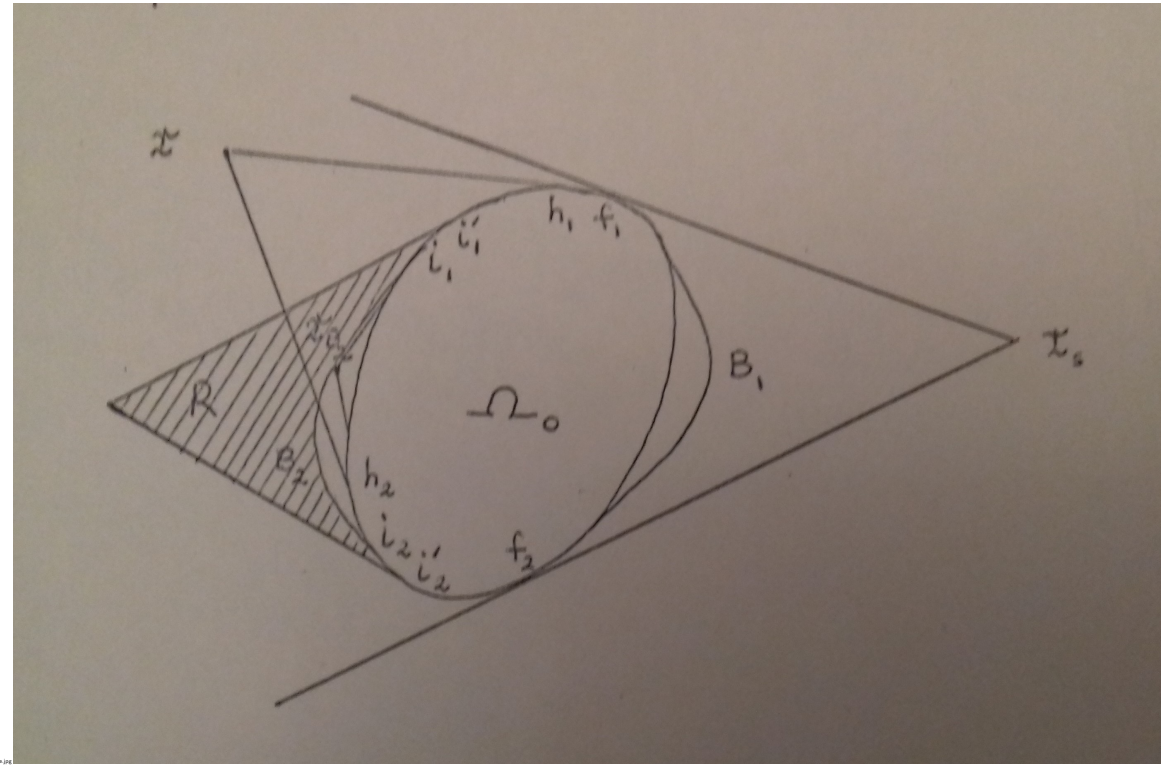


Figure : Perturbed ellipse

Validity of the Geometrical Theory of Diffraction

Theorem (BDS 1976)

As $k \to \infty$

$$U(\mathbf{r}; \mathbf{r_s}; k) = U_{ellipse}(\mathbf{r}; \mathbf{r_s}; k) \left[1 + \mathcal{O}(\exp^{-k^{1/3}\mu})\right]$$

uniformly in $\mathbf{r}$, $\mathbf{r} \epsilon S_2^{<}(\mathbf{r_s}) - R$, *where* μ *is positive and independent of* k *and* $\mathbf{r}$, *where* $S_2^{<}(\mathbf{r_s})$ *is the "deep shadow" of* Ω_2 *and* R *is the "region of influence" of* B_2.

A 3D analogue of this result is needed and may depend on full high frequency analysis of scattering by a prolate spheroid.

Inverse problem of Obstacle scattering

Keller (1976): What is the question to which the answer is Nine W?

D a bounded, simply connected domain of class C^2 with boundary ∂D.

ν the inward unit normal to ∂D.

$u^i(x,k)$ is the incident field and $u_\infty(\hat{x})$ is the far field pattern. Let $\Phi(x,y)$ be the fundamental solution of the Helmholtz equation.

Given knowledge of the far field pattern determine the unknown scattering obstacle D.

An early attempt at a solution is due to Imbriale and Mittra (1970),BDS(1982). Here they fix a coordinate origin, calculate the scattered field from $u_\infty(\hat{x})$ and then use least squares to find points where the total field vanishes. Choose another origin of coordinates and repeat the process until a sufficient number of surface points have been obtained to ascertain the shape of D.

Inverse problem of Obstacle scattering

At high frequency Keller (1959) used a physical optics approximation which, except in special cases, rests on the solution of Minkowski's problem. Is an obstacle uniquely determined by surface Gaussian Curvature?
Huge progress since (1970) c.f.Colton Kress (2013).
We have the field representations

$$u^i(x,k) = -\int_{\partial D} \Phi(x,y)\frac{\partial u}{\partial \nu} ds(y),\ x \in \partial D, \tag{5}$$

$$u_\infty(\hat{x},k) = \gamma \int_{\partial D} e^{-ik\hat{x}\cdot y}\frac{\partial u}{\partial \nu} ds(y). \tag{6}$$

This is the basis of a Newton type method BDS (1982), Johansson and BDS (2007).
Need for a rigorous convergence theory.

Inverse problem of Obstacle scattering

Theorem (Schiffer 1967,Lax and Philips (1967))

Assume D_1 and D_2 are two sound soft scatterers such that the far field patterns coincide for an infinite number of incident plane waves with distinct directions and one fixed wave number. Then $D_1 = D_2$.

Theorem (Colton and BDS 1983)

Let D_1 and D_2 be two sound soft scatterers which are contained in a ball of radius R such that $kR < \pi$ and assume that the far field patterns coincide for one incident plane wave with wave number k. Then $D_1 = D_2$.

Problems Does the far field pattern for scattering of one incident plane wave at one single wave number determine the scatterer? What can be said about sound hard or impedance scatterers?

Nagumo Equation

H. P. McKean (1970)
Nagumo's equation is a starting point for modelling and studying nerve impulse propagation in unmyelinated nerve axons.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u)(u-a).$$

Travelling waves

$$u(x,\ t) = \phi(x + ct),$$

Unique stable heteroclinic orbit in which , with $\xi = x + ct$,

$$\phi(\xi) = \frac{1}{1 + \exp(-\xi/\sqrt{2})},\ c = \sqrt{2}(\frac{1}{2} - a)$$

McKean gives full phase plane analysis. model does not account for recovery to a resting state.

Nerve Impulse Propagation

Recovery is achieved with the FitzHugh-Nagumo Model

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + u(1-u)(u-a) - w, \\ \frac{\partial w}{\partial t} &= bu - \gamma w. \end{aligned}$$

Look for travelling waves $u(x,\ t) = \phi(x + ct)$ and $w(x,\ t) = \psi(x + ct)$
Need 3D phase space analysis

Nerve Impulse Propagation

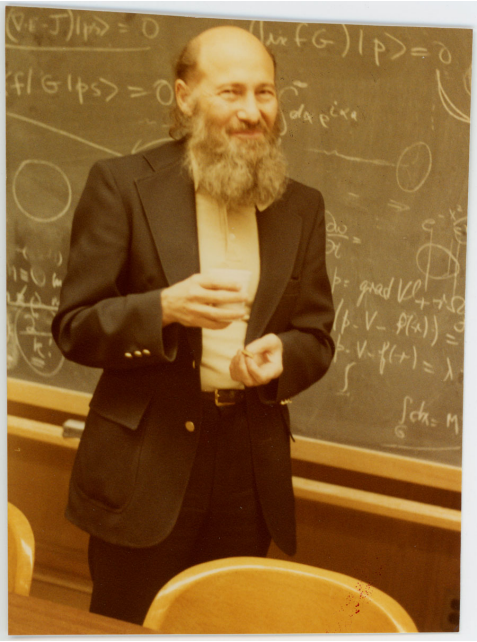
Rinzel and Keller (1973) simplified matters by modelling the cubic in u by

$$F(u) = -u + H(u-a), \quad \text{H -Heaviside step function}$$

Later BDS (see Jones, Plank and BDS (2010)) treated the more general form

$$\begin{aligned} F(u) &= -u, \ u \leq a/2, \\ &= u - a, \ a/2 < u \leq (1+a)/2, \\ &= 1 - u, \ (1+a)/2 \leq u. \end{aligned}$$

Open Problems: evolutionary structure of the F-N PDE; Threshold behaviour; more on stability of travelling waves. What about mylinated fibres (essential to mammals)?



Thank you Joe for so much joy and inspiration.