„On Nonreflecting Boundary Conditions“

G.-Keller, J. Comp. Phys., 1995

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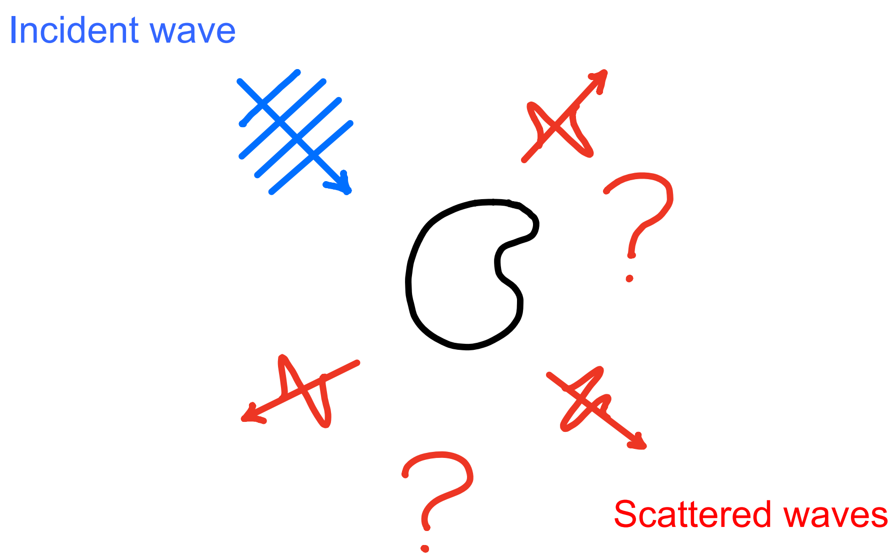
Unbounded domains

GOAL: simulate waves in unbounded domains,
without spurious reflection from artificial boundaries



How do we simulate an infinite domain…
...on a finite computer ???

Unbounded domains

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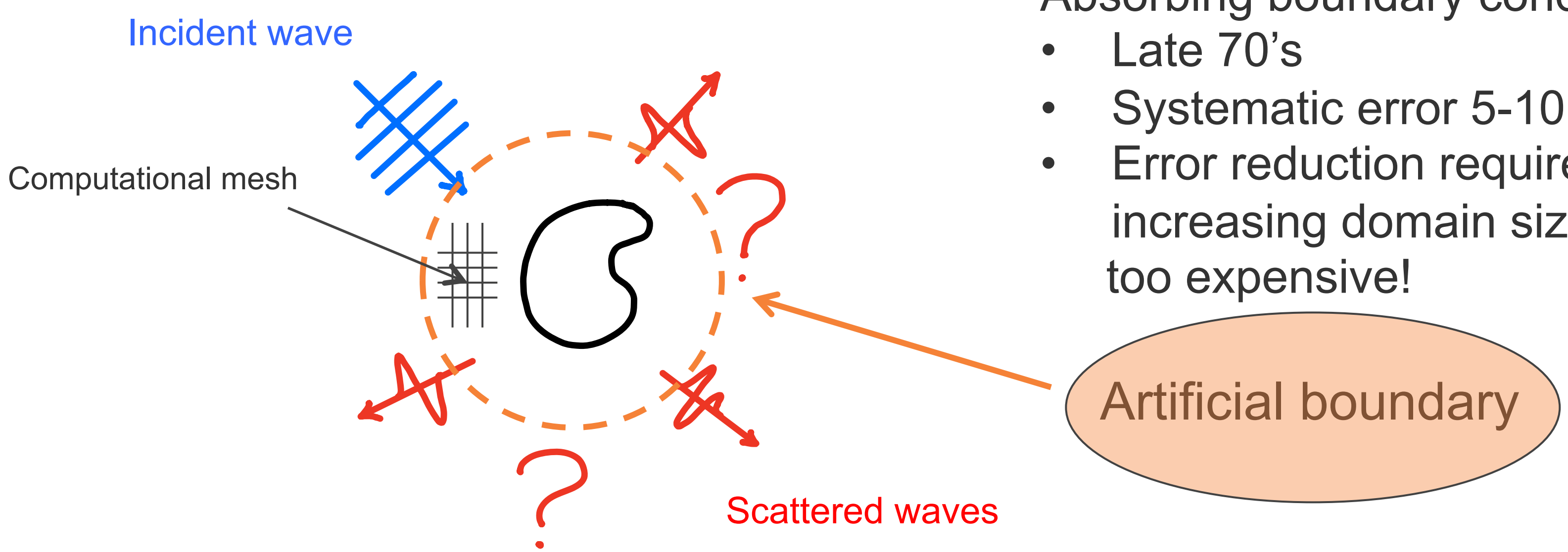


Absorbing boundary conditions

- Late 70's
- Systematic error 5-10%
- Error reduction requires increasing domain size, too expensive!

Time-harmonic scattering (2D)

In the exterior, the scattered field satisfies the Helmholtz equation with wave number k and the Sommerfeld radiation condition:

$$\begin{aligned} \Delta u + k^2 u &= 0 \quad \text{in } D, \qquad k > 0 \text{ constant}, \\ \lim_{r\to\infty} \sqrt{r}\left(\frac{\partial}{\partial r} - ik\right) u &= 0, \qquad r := |\boldsymbol{x}|. \end{aligned}$$

The solution is uniquely determined by its values on a circle of radius R:

$$u(r,\theta) = \frac{1}{\pi} {\sum_{n=0}^{\infty}}' \frac{H_n^{(1)}(kr)}{H_n^{(1)}(kR)} \int_0^{2\pi} u(R,\theta') \cos n(\theta - \theta')\, d\theta', \qquad r \geq R.$$

Dirichlet-to-Neumann (DtN) map

Differentiate w.r.t. r and set r = R

$$\frac{\partial u}{\partial r}(R,\theta) = {\sum_{n=0}^{\infty}}' \frac{k}{\pi} \frac{H_n^{(1)'}(kR)}{H_n^{(1)}(kR)} \int_0^{2\pi} u(R,\theta') \cos n(\theta - \theta')\, d\theta'$$

This is the (exact) Dirichlet-to-Neumann boundary condition:

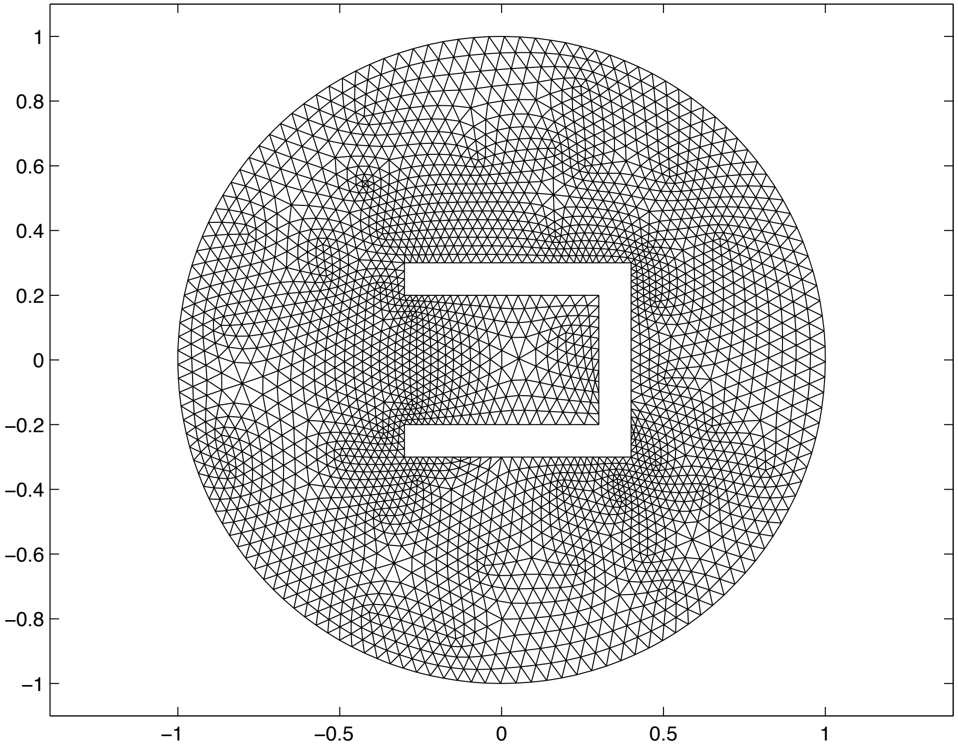
$$\frac{\partial u}{\partial n} = Mu \quad \text{on } B$$

J.B. Keller and D. Givoli (1989), …

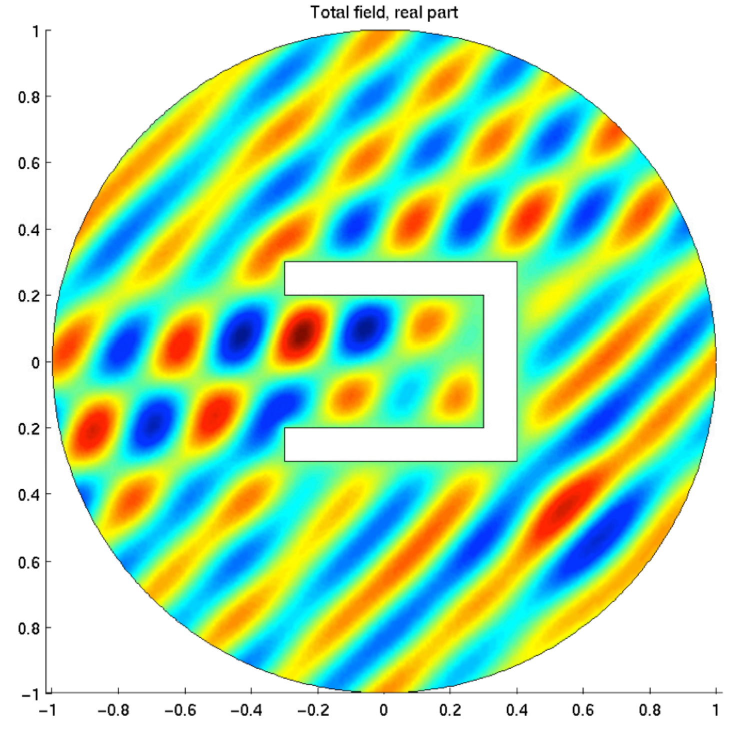
DtN-FE Method: open cavity (2D)

Incident plane wave

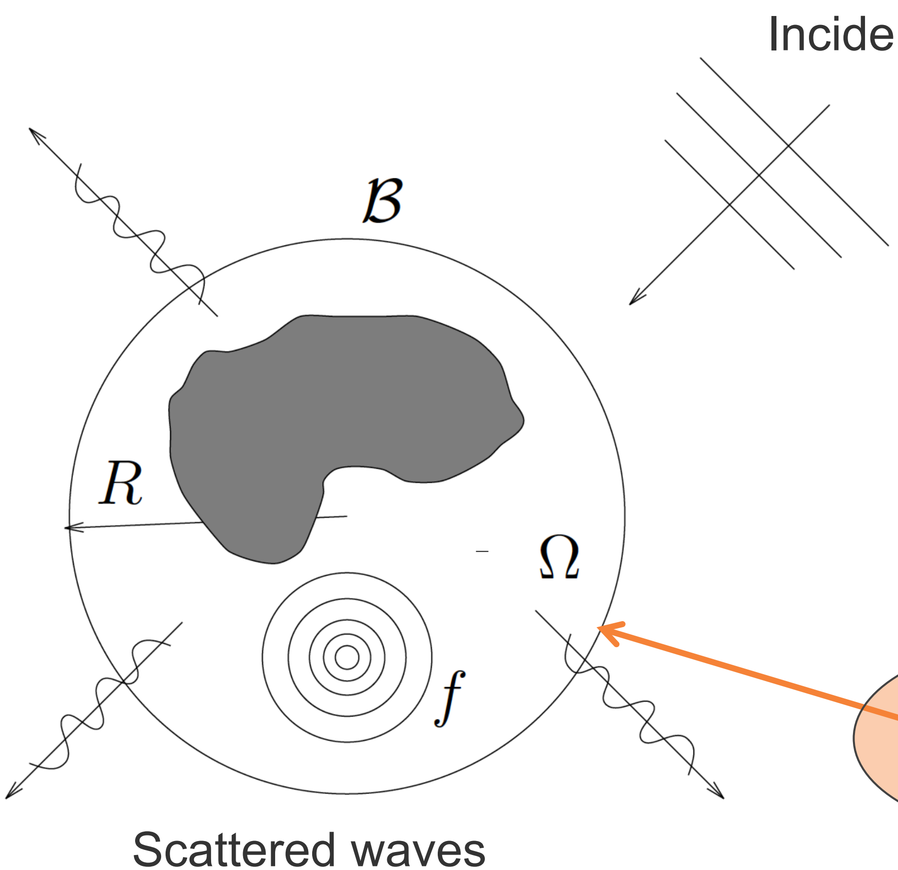
FE mesh



Numerical solution: total wave field



Time dependent scattering (3D)



Outside, the wave speed c is constant:

$$u_{tt} - c^2 \Delta u = 0,$$
$$u(\mathbf{x}, 0) = 0, \quad u_t(\mathbf{x}, 0) = 0.$$

Artificial boundary

Hence, all waves are purely outgoing

Artificial boundary conditions: 90's

State-of-the-art in the early 90's:

- Local absorbing boundary conditions (ABC's): Engquist-Majda (1977), Bayliss-Turkel (1980), …
  easy, cheap, systematic 5-10% error from spurious reflection

- Global, based on Kirchhoff formula: Ting-Miksis (1980), …
  exact, but very expensive, needs entire space-time history
  on the artificial boundary

Question: Is there an exact local nonreflecting boundary condition for the wave equation ?

Nonreflecting Boundary Condition

Fourier decomposition in the exterior:

$$u(r,\theta,\varphi,t) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} u_{nm}(r,t) Y_{nm}(\theta,\varphi).$$

Normal incidence (n = 0): $\quad u = u(r,t)$

$$(ru)_{tt} - (ru)_{rr} = 0, \qquad r > R.$$

$$\Rightarrow \quad ru(r,t) = f(r-t) + g(r+t).$$

Exact and local NBC:

$$(\partial_r + \partial_t)\,[ru] = 0, \qquad r = R.$$

Nonreflecting Boundary Condition

General case, n ≥ 1:

$$(\partial_{tt} - \partial_{rr})\, G_n[u_{nm}] = 0, \qquad r > R.$$

$$(\partial_r + \partial_t)\, G_n[u_{nm}] = 0, \qquad r = R.$$

NBC, exact and local:

$$\frac{\partial u_{nm}}{\partial r} + \frac{\partial u_{nm}}{\partial t} + \frac{u_{nm}}{R} = -\frac{1}{R^2}\, \boldsymbol{d}_n \cdot \boldsymbol{\psi}_{nm}(t), \qquad r = R.$$

$$\frac{d}{dt}\boldsymbol{\psi}_{nm}(t) = \frac{1}{R}\, \boldsymbol{A}_n \boldsymbol{\psi}_{nm}(t) + u_{nm}(R,t)\, \boldsymbol{e}_n, \qquad \boldsymbol{\psi}_{nm}(0) = 0.$$

multiply by Y_{nm} and sum over n and m...

Nonreflecting Boundary Condition

$$\frac{\partial u}{\partial r} + \frac{\partial u}{\partial t} + \frac{u}{R} = -\frac{1}{R^2} \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \boldsymbol{d}_n \cdot \boldsymbol{\psi}_{nm}(t)\, Y_{nm}(\vartheta, \phi), \qquad r = R,$$

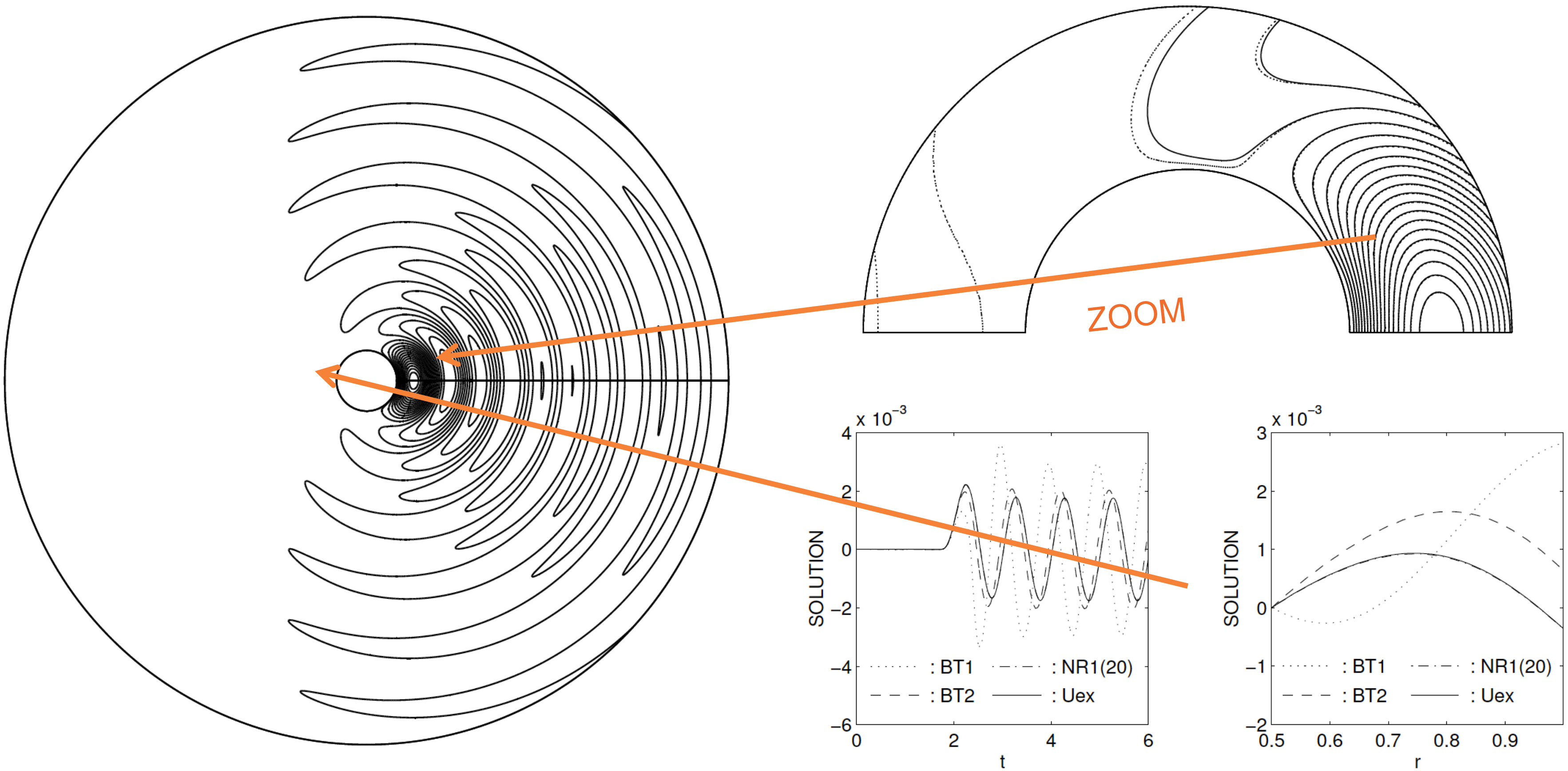
$$\frac{d}{dt}\boldsymbol{\psi}_{nm}(t) = \frac{1}{R}\, \boldsymbol{A}_n \boldsymbol{\psi}_{nm}(t) + (u|_{r=R}\,, Y_{nm})\, \boldsymbol{e}_n, \qquad \boldsymbol{\psi}_{nm}(0) = 0.$$

$\boldsymbol{e}_n$ and $\boldsymbol{d}_n$ are constant n-vectors
$\boldsymbol{A}_n$ are constant $n \times n$ matrices with eigenvalues in $\mathbb{C}_-$

- local in time, only first-order derivatives
- fits easily into FD/FE methods
- in practice sum truncated: $n \le N$, $N \simeq 20$.
- storage $\sim 2N^3/3$ scalar values for $\boldsymbol{\psi}_{nm}$

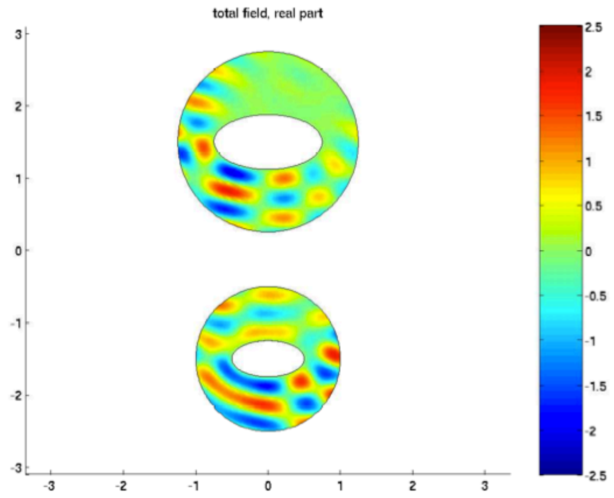
G.-Keller, SIAM Appl. Math. (1995)

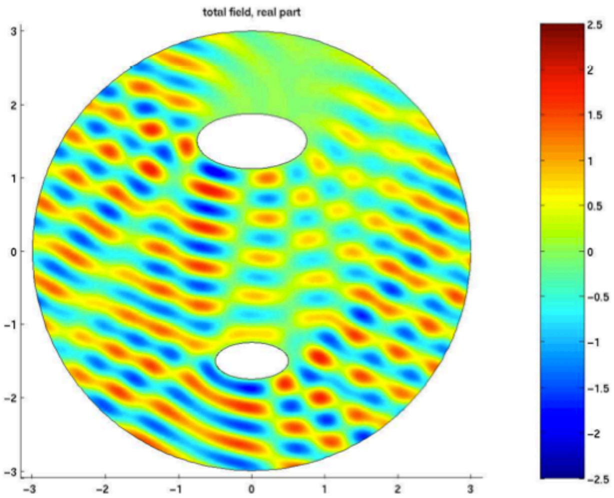
Numerical example: piston on sphere



NBC's: beyond the mid 90's

- Extended to Maxwell's Equations and elasticity: (G.-Keller, 1998, 2000), …
- Renewed general interest in exact NBC's: Hagstrom, Hariharan (1998); Alpert, Greengard, Hagstrom (2000); Lubich, Schädle (2002); …
- Paved the way for space-time local NBC's: HH (1998); G. (2006);
- Extended to multiple scattering: (G.-Kirsch, 2004), …





Concluding Remarks

- Joe Keller’s work also had a profound impact on computational wave propagation
- Joe introduced me to the “world of waves”
- Working with Joe was a huge privilege

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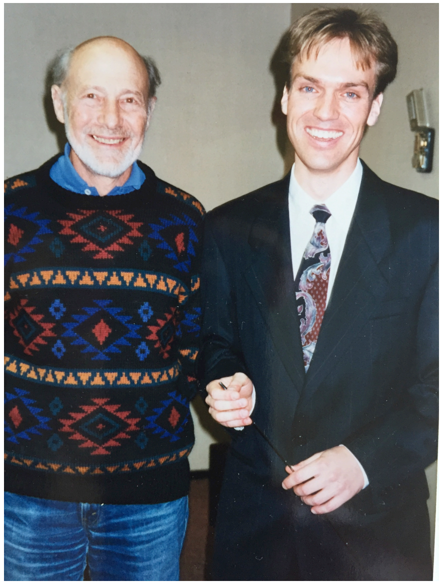

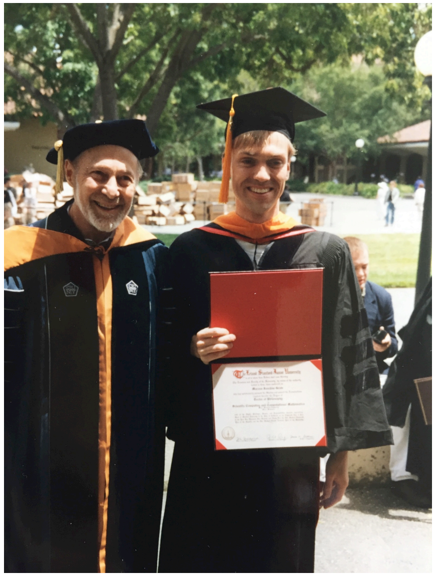

Stanford University, 1995