Kellerods and Kelasticas

Alain Goriely, Oxford

Part 1: Reading Joe Keller’s work on elastic rods

Part 2: Geometric stability methods for 1-D problems
Keller’s contributions to rods

- About 18 publications from 1951 to 2010
- 1951: Bowing of Violin Strings
- 2010: Ponytail motion
Keller’s contributions to rods

- About 18 publications from 1951 to 2010

- 1951: Bowing of Violin Strings
- 2010: Ponytail motion

- Roughly divided into
  - Wave propagation
  - Contact problems
  - Optimal/Ideal shapes and Stability
Contact problems

• Post buckling behavior of elastic tubes and rings with opposite sides in contact (1972, with Flaherty & Rubinow)
• Contact problems involving a buckled elastica (1973, with Flaherty)
• Some bubble and contact problems (1980)
Contact problems

Stability of two rings

With Gaetano Napoli Tom Mullin

Grotberg & Jensen 2004 (Heil)
Contact problems

- Ropes in equilibrium (1987, with Maddocks)
Optimal/Ideal shapes

- Strongest column (1960)
- Tallest column (1966, with Niordson)
- Rope of minimum elongation (1984, with Verma)

“To find the curve which by its revolution around an axis determines the column of greatest efficiency”

Clamped

Hinged

Cox 1992
Optimal/Ideal shapes

• Strongest column (1960)
• Tallest column (1966, with Niordson)
• Rope of minimum elongation (1984, with Verma)
Optimal/Ideal shapes

• Strongest column (1960)
• Tallest column (1966, with Niordson)
• Rope of minimum elongation  (1984, with Verma)
Optimal/Ideal shapes

• Tendril shape (1984)
Optimal/Ideal shapes

• Mobius band (1993, with Mahadevan)
• Coiling of ropes (1995, with Mahadevan)

Starostin-van der Heijden, 2007
Stability methods for one-dimensional problems (work with Th. Lessinnes)
Dynamics in phase space
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More generally
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For initial value problems, the geometry of curves in phase space provides information on the stability of solutions.
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For boundary value problems, equilibrium solutions also lie in phase space.
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For boundary value problems, equilibrium solutions also lie in phase space.

Stability?
Example: hanging rod

\[ E = \int_0^1 \frac{(\theta' - \sqrt{2Mv})^2}{2} + M \cos \theta \, dx, \quad V = -M \cos \theta. \]

\[ \theta'' + M \sin \theta = 0, \quad \theta'(0) = \theta'(1) = \sqrt{2Mv}. \]

Take for instance:

\( v = 1.5 \) and \( M = 81 \)
Problem statement

Define the functional

$$\mathcal{E}[\theta] = \int_a^b \mathcal{L} \left( \theta(s), \theta'(s) \right) \, ds$$

with

$$\mathcal{L}(\theta, \theta') = \frac{(\theta' - A)^2}{2} - V(\theta),$$

$$\theta(a) = \theta_a, \quad \theta(b) = \theta_b$$

or

$$\theta'(a) = \theta'(b) = A.$$

Find a function $\theta$ which locally minimises $\mathcal{E}$: $\mathcal{E}[\theta + \epsilon \tau] > \mathcal{E}[\theta]$
Problem statement

Define the functional

\[ \mathcal{E}[\theta] = \int_a^b \mathcal{L} \left( \theta(s), \theta'(s) \right) \, ds \]

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\[ \theta'(a) = \theta'(b) = A. \]

Find a function \( \theta \) which locally minimises \( \mathcal{E} : \quad \mathcal{E}[\theta + \epsilon \tau] > \mathcal{E}[\theta] \)

To first order in \( \epsilon \),

\[ \delta \mathcal{E}_\theta(\tau) = 0 \quad \forall \tau \in C'([a, b]) \quad \Rightarrow \quad \frac{\partial \mathcal{L}}{\partial \theta} - \frac{d}{ds} \frac{\partial \mathcal{L}}{\partial \theta'} = 0. \]
Problem statement

Define the functional

$$\mathcal{E} [\theta] = \int_a^b \mathcal{L} \left( \theta(s), \theta'(s) \right) \, ds$$

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Find a function $\theta$ which locally minimises $\mathcal{E}$: $\mathcal{E}[\theta + \epsilon \tau] > \mathcal{E}[\theta]$.

To first order in $\epsilon$,

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To second order:

$$\delta^2 \mathcal{E}_{\theta}(\tau) > 0 \quad \forall \tau \in C'([a, b])$$
Problem statement

Define the functional

\[ E[\theta] = \int_a^b L(\theta(s), \theta'(s)) \, ds \]

with

\[ L(\theta, \theta') = \frac{(\theta' - A)^2}{2} - V(\theta), \quad \theta(a) = \theta_a, \quad \theta(b) = \theta_b \]

\[ \theta'(a) = \theta'(b) = A. \]

Find a function \( \theta \) which locally minimises \( E \):

\[ E[\theta + \epsilon \tau] > E[\theta] \]

To first order in \( \epsilon \),

\[ \delta E_\theta(\tau) = 0 \quad \forall \tau \in C'(\mathbb{R}) \quad \Rightarrow \quad \frac{\partial L}{\partial \theta} - \frac{d}{ds} \frac{\partial L}{\partial \theta'} = 0. \]

To second order:

\[ \delta^2 E_\theta(\tau) > 0 \quad \forall \tau \in C'(\mathbb{R}) \]

\[ C'(\mathbb{R}) \equiv \{ f \in C^1([a, b]) \setminus \{0\} : f'(a) = f'(b) = 0 \} \]
**Theorem 2** [Lessinnes&AG, 2017]: Consider a stationary solution of

\[
\mathcal{E}[\theta] = \int_a^b \frac{(\theta'(s) - A)^2}{2} - V(\theta(s)) \, ds \quad \theta'(a) = \theta'(b) = A.
\]

and its corresponding solution in the phase plane \( \gamma : s \in [a, b] \to (\theta(s), \theta'(s)) \)
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Define \( I = \) the number of max crossing minus the number of min crossing

\[ = \text{number of summits} - \text{number of valleys} \]

\[ = \text{number of blue lines crossed} - \text{number of red lines crossed} \]
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Define \( I = \) the number of max crossing minus the number of min crossing

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Then, if \( I > 0 \), the second variation is strictly positive (local minimum=stable)

if \( I < 0 \), the second variation is not positive (not a minimum=unstable)
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If \( I = 0 \), Define \( J \) as the weighted difference in slopes at the end points

\[ J = A(V'(\theta(b)) - V'(\theta(a))) \]
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If \( I = 0 \), Define \( J \) as the weighted difference in slopes at the end points

\[
J = A(V'(\theta(b)) - V'(\theta(a))
\]

Then, if \( J \leq 0 \), unstable

if \( J > 0 \), ? numerical
Example: hanging rod

Stability: Count the number of max-number of min

One max, no min $\Rightarrow I > 0 \Rightarrow$ Stable
I = -1
Unstable

K = 2
Dirichlet
unstable
$I = -1$
Unstable

$K = 2$
Dirichlet
unstable

$I = 1$
Stable

$K = 0$
Dirichlet stable
\( I = 1 \)
Stable

\( K = 2 \)
Dirichlet unstable

\( I = 0 \)
but \( J = 0 \)
Unstable

\( K = 0 \)
Dirichlet stable
I = -1
Unstable
K = 2
Dirichlet
unstable

I = 1
Stable
K = 0
Dirichlet stable

I = 0
but J = 0
Unstable
K = 0
Dirichlet stable

2 more stable cases and 2 more unstable ones (not shown)
Conclusions

- I wish I were as creative as Keller

Geometric Stability (Nonlinearity- 2017)  TL, AG
Stability of birods (SIAM J. Appl Math- 2016), TL, AG
Conclusions

- I wish I were as creative as Keller
- If you can draw a solution, you know its stability

Growing rods

Bi-rods

Thomas Lessinnes

Geometric Stability (Nonlinearity- 2017)  TL, AG
Stability of birods (SIAM J. Appl Math- 2016), TL, AG
The second variation

\[ \delta^2 \mathcal{E}_\theta(\tau) = \int_a^b (\tau')^2 - \frac{d^2V}{d\theta^2} \bigg|_{\theta(s)} \tau^2 \; ds > 0 \quad \forall \tau \in \mathcal{C}'([a, b]) \]

\[
\mathcal{L}(\theta, \theta') = \frac{(\theta' - A)^2}{2} - V(\theta)
\]

\[
\begin{cases}
\theta'' + \frac{dV}{d\theta} = 0 \\
\theta'(a) = \theta'(b) = A.
\end{cases}
\]
The second variation

\[
\delta^2 \mathcal{E}_\theta(\tau) = \int_a^b (\tau')^2 \left. - \frac{d^2V}{d\theta^2} \right|_{\theta(s)} \tau^2 \, ds > 0 \quad \forall \tau \in \mathcal{C}'([a, b])
\]

\[
= \langle \tau | \mathcal{S}_\tau \rangle > 0
\]

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\mathcal{L}(\theta, \theta') = \frac{(\theta' - A)^2}{2} - V(\theta)
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The second variation

$$\delta^2 \mathcal{E}_\theta(\tau) = \int_a^b (\tau')^2 \left. \frac{d^2 V}{d\theta^2} \right|_{\theta(s)} \tau^2 \ ds > 0 \ \forall \tau \in C'([a, b])$$

$$= \langle \tau | \mathcal{S} \tau \rangle > 0$$

$$\mathcal{S} = -\frac{d^2}{ds^2} + f(s) \quad f(s) = -\left. \frac{d^2 V}{d\theta^2} \right|_{\theta(s)}$$

$$\mathcal{L}(\theta, \theta') = \frac{(\theta' - A)^2}{2} - V(\theta)$$

$$\begin{cases}
\theta'' + \frac{dV}{d\theta} = 0 \\
\theta'(a) = \theta'(b) = A.
\end{cases}$$
The second variation

\[ \delta^2 \mathcal{E}_\theta(\tau) = \int_a^b (\tau')^2 - \left. \frac{d^2 V}{d\theta^2} \right|_{\theta(s)} \tau^2 \, ds > 0 \quad \forall \tau \in \mathcal{C}'([a, b]) \]

\[ = \langle \tau \mid \mathcal{S}\tau \rangle > 0 \]

\[ \mathcal{S} = -\frac{d^2}{ds^2} + f(s) \quad f(s) = -\left. \frac{d^2 V}{d\theta^2} \right|_{\theta(s)} \]

Sturm-Liouville problem.

\[ \mathcal{S}\tau = \lambda\tau \]

\[ \tau'(a) = \tau'(b) = 0. \]

Stability iff \( \lambda > 0 \)
The second variation

\[ \delta^2 \mathcal{E}_\theta(\tau) = \int_a^b (\tau')^2 - \frac{d^2V}{d\theta^2} \bigg|_{\theta(s)} \tau^2 \ ds > 0 \quad \forall \tau \in \mathcal{C}'([a, b]) \]

\[ = \langle \tau | S \tau \rangle > 0 \quad \forall \tau \in \mathcal{D}'([a, b]) \]

\[ S = -\frac{d^2}{ds^2} + f(s) \quad \quad f(s) = -\frac{d^2V}{d\theta^2} \bigg|_{\theta(s)} \]

Sturm-Liouville problem.

\[ \mathcal{L}(\theta, \theta') = \frac{(\theta' - A)^2}{2} - V(\theta) \]

\[ \begin{cases} \theta'' + \frac{dV}{d\theta} = 0 \\ \theta'(a) = \theta'(b) = A. \end{cases} \]

Stability iff \( \lambda > 0 \)
The second variation

\[\delta^2 \mathcal{E}_{\theta}(\tau) = \int_a^b (\tau')^2 - \frac{d^2 V}{d\theta^2} \bigg|_{\theta(s)} \tau^2 \ ds > 0 \ \forall \tau \in \mathcal{C}'([a, b])\]

\[= \langle \tau | S \tau \rangle > 0 \ \forall \tau \in \mathcal{D}'([a, b])\]

\[S = -\frac{d^2}{ds^2} + f(s) \quad f(s) = -\frac{d^2 V}{d\theta^2} \bigg|_{\theta(s)}\]

\[\mathcal{L}(\theta, \theta') = \frac{(\theta' - A)^2}{2} - V(\theta)\]

\[\begin{cases} 
\theta'' + \frac{dV}{d\theta} = 0 \\
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\end{cases}\]

Sturm-Liouville problem.

\[S \tau = \lambda \tau \]

\[\tau'(a) = \tau'(b) = 0.\]

Stability iff \(\lambda > 0\)

\[\sigma \quad \text{First auxiliary problem}\]

\[S \tau = \lambda \tau \]

\[\tau'(a) = \tau'(\sigma) = 0.\]
The second variation

\[ \delta^2 \mathcal{S}_\theta(\tau) = \int_a^b (\tau')^2 - \frac{d^2V}{d\theta^2}\bigg|_{\theta(s)} \tau^2 \ ds > 0 \ \forall \tau \in C'(\mathbb{R}) \]

\[ = \langle \tau | \mathcal{S} \tau \rangle > 0 \ \forall \tau \in \mathcal{D}'(\mathbb{R}) \]

\[ \mathcal{S} = -\frac{d^2}{ds^2} + f(s) \quad f(s) = -\frac{d^2V}{d\theta^2}\bigg|_{\theta(s)} \]

Sturm-Liouville problem.

\[ \mathcal{L}(\theta, \theta') = \frac{(\theta' - A)^2}{2} - V(\theta) \]

\[ \begin{cases} 
\theta'' + \frac{dV}{d\theta} = 0 \\
\theta'(a) = \theta'(b) = A.
\end{cases} \]

Stability iff \( \lambda > 0 \)

\[ \tau'(a) = \tau'(b) = 0. \]

First auxiliary problem

\[ \mathcal{S}_\tau = \lambda \tau \]

\[ \tau'(a) = \tau'(\sigma) = 0. \]
The second variation

$$\delta^2 \mathcal{E}_\theta(\tau) = \int_a^b (\tau')^2 \left. - \frac{d^2V}{d\tau^2}\right|_{\theta(s)} \tau^2 \, ds > 0 \quad \forall \tau \in C'(\mathbb{R})$$

$$= \langle \tau | S \tau \rangle > 0 \quad \forall \tau \in D'(\mathbb{R})$$

$$S = -\frac{d^2}{ds^2} + f(s) \quad f(s) = -\left. \frac{d^2V}{d\tau^2}\right|_{\theta(s)}$$

Sturm-Liouville problem.

$$\mathcal{L}(\theta, \theta') = \frac{(\theta' - A)^2}{2} - V(\theta)$$

$$\begin{cases} 
\theta'' + \frac{dV}{d\theta} = 0 \\
\theta'(a) = \theta'(b) = A.
\end{cases}$$

Stability iff $\lambda > 0$

First auxiliary problem

$$S\tau = \lambda \tau$$

$$\tau'(a) = \tau'(b) = 0.$$
The second variation

\[ \delta^2 \mathcal{E}_\theta(\tau) = \int_a^b (\tau')^2 - \frac{d^2V}{d\theta^2} \bigg|_{\theta(s)} \, ds > 0 \quad \forall \tau \in C'(\[a, b]) \]

\[ = \langle \tau | S \tau \rangle > 0 \quad \forall \tau \in D'(\[a, b]) \]

\[ S = -\frac{d^2}{ds^2} + f(s) \quad f(s) = -\frac{d^2V}{d\theta^2} \bigg|_{\theta(s)} \]

Sturm-Liouville problem.

\[ \mathcal{L}(\theta, \theta') = \frac{(\theta' - A)^2}{2} - V(\theta) \]

\[ \begin{cases} 
\theta'' + \frac{dV}{d\theta} = 0 \\
\theta'(a) = \theta'(b) = A. 
\end{cases} \]

Stability iff \( \lambda > 0 \)

First auxiliary problem

\[ S \tau = \lambda \tau \]

\[ \tau'(a) = \tau'(b) = 0. \]

Conjugate point to \( a \)
Sturm-Liouville problem.

\[ S \tau = \lambda \tau \]
\[ \tau'(a) = \tau'(b) = 0. \]

Stability iff \( \lambda > 0 \)

First auxiliary problem

\[ S \tau = \lambda \tau \]
\[ \tau'(a) = \tau'(\sigma) = 0. \]
First auxiliary problem
\[ \mathcal{S}\tau = \lambda \tau \]
\[ \tau'(a) = \tau'(\sigma) = 0. \]

Second auxiliary problem
\[ \mathcal{S}h = 0, \]
\[ h(a) = 1, \quad h'(a) = 0. \]

Stability iff \( \lambda > 0 \)
Sturm-Liouville problem.

\[ S\tau = \lambda \tau \]

\[ \tau'(a) = \tau'(b) = 0. \]

Stability iff \( \lambda > 0 \)

First auxiliary problem

\[ S\tau = \lambda \tau \]

\[ \tau'(a) = \tau'(\sigma) = 0. \]

Second auxiliary problem

\[ Sh = 0, \]

\[ h(a) = 1, \ h'(a) = 0 \]

\[ \Rightarrow \text{The roots of } h' \text{ are the conjugate points} \]
First auxiliary problem
\[ S_T = \lambda T \]
\[ \tau'(a) = [\tau'(\sigma)] = 0. \]

Second auxiliary problem
\[ S h = 0, \]
\[ h(a) = 1, \quad h'(a) = 0. \]

⇒ The roots of \( h' \) are the conjugate points
⇔ The roots of \( \theta'' \) are the conjugate points

Stability iff \( \lambda > 0 \)

Sturm-Liouville problem.
\[ S_T = \lambda T \]
\[ \tau'(a) = \tau'(b) = 0. \]
First auxiliary problem
\[ S\tau = \lambda \tau \]
\[ \tau'(a) = [\tau'(\sigma)] = 0. \]

Second auxiliary problem
\[ Sh = 0, \]
\[ h(a) = 1, \quad h'(a) = 0 \]

→ The roots of \( h' \) are the conjugate points
⇔ The roots of \( \theta'' \) are the conjugate points
⇔ The extrema of \( V \) give the conjugate points

Conjugate point to a

Stability iff \( \lambda > 0 \)

Sturm-Liouville problem.
\[ S\tau = \lambda \tau \]
\[ \tau'(a) = \tau'(b) = 0. \]