Kellerods and Kelasticas

Alain Goriely, Oxford

Part 1: Reading Joe Keller's work on elastic rods

Part 2: Geometric stability methods for 1-D problems

Keller’s contributions to rods

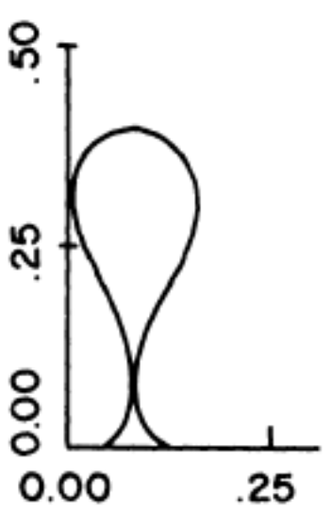
- About 18 publications from 1951 to 2010
- 1951:Bowing of Violin Strings
- 2010:Ponytail motion

Keller’s contributions to rods

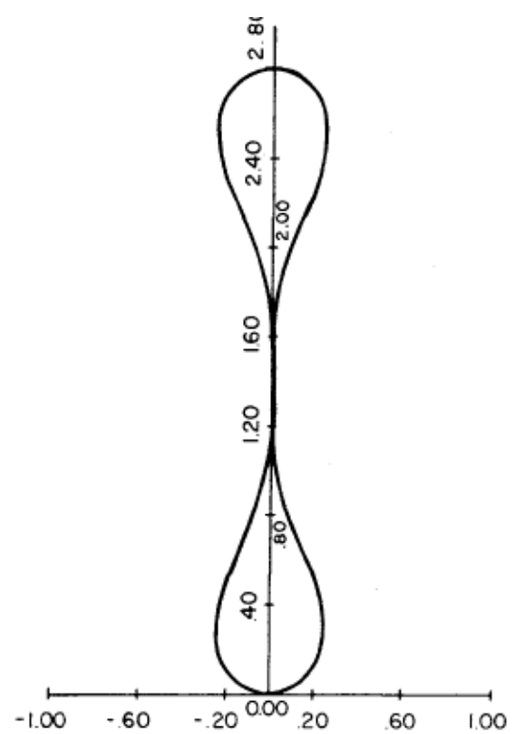
- About 18 publications from 1951 to 2010
- 1951:Bowing of Violin Strings
- 2010:Ponytail motion
- Roughly divided into
  - Wave propagation
  - Contact problems
  - Optimal/Ideal shapes and Stability

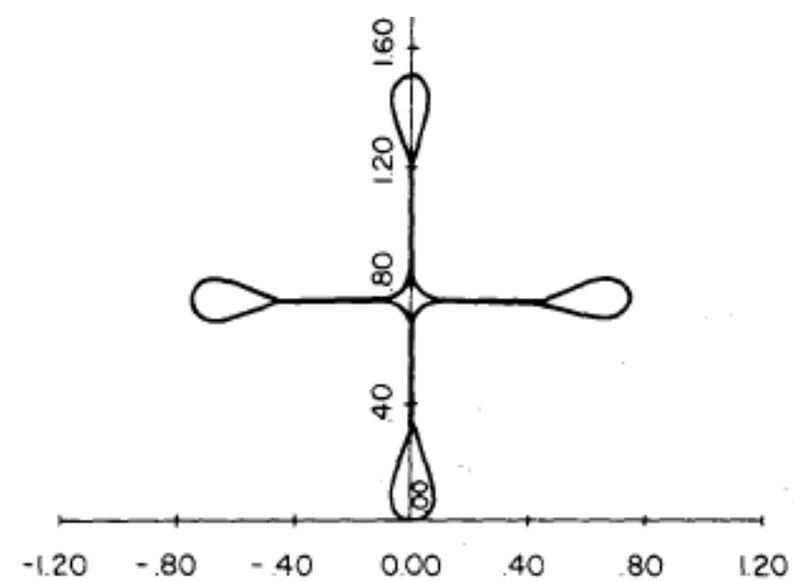
Contact problems

- Post buckling behavior of elastic tubes and rings with opposite sides in contact (1972, with Flaherty & Rubinow)
- Contact problems involving a buckled elastica (1973, with Flaherty)
- Some bubble and contact problems (1980)



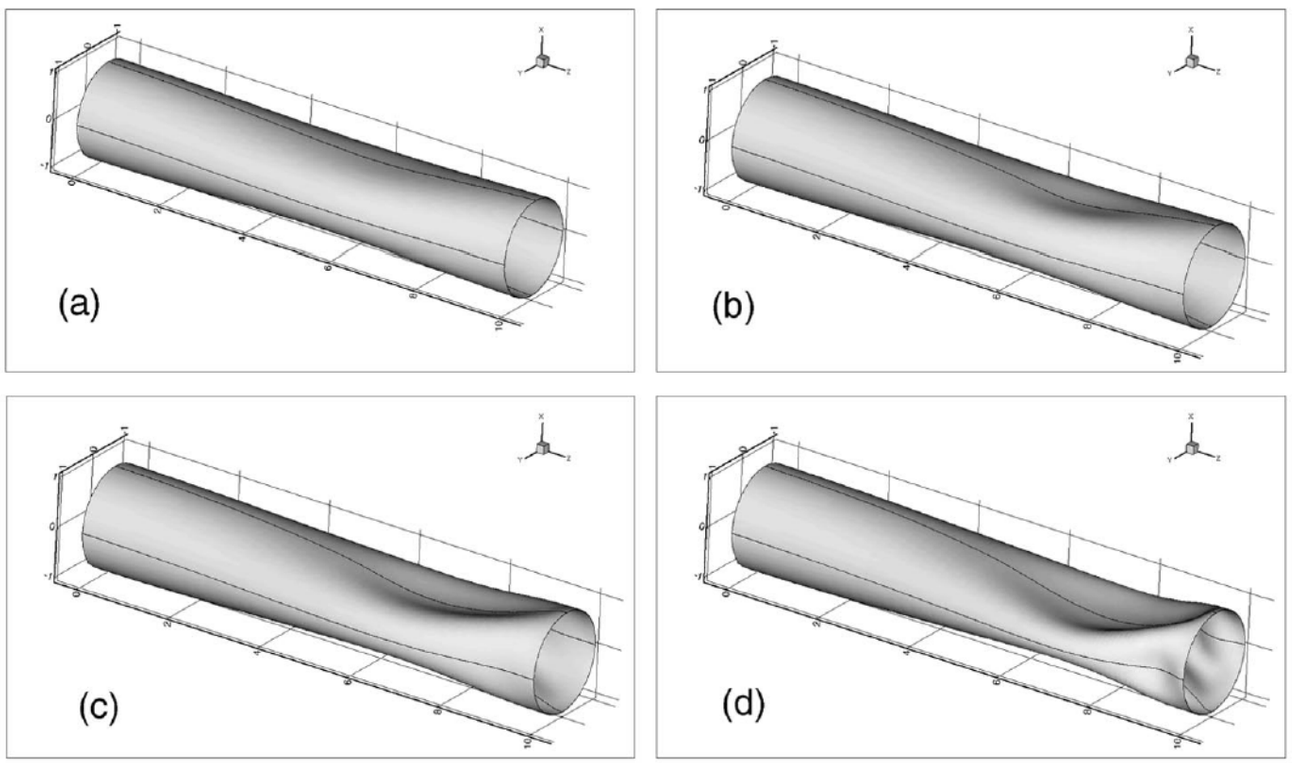
P= 72.187
$s_1 = 0.130$





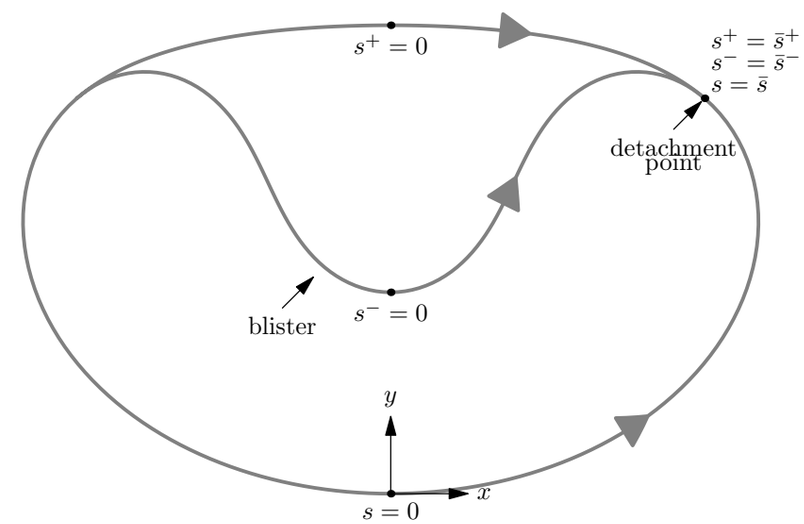
P=500.000
S_1 =.431

Contact problems



Grotberg & Jensen 2004 (Heil)

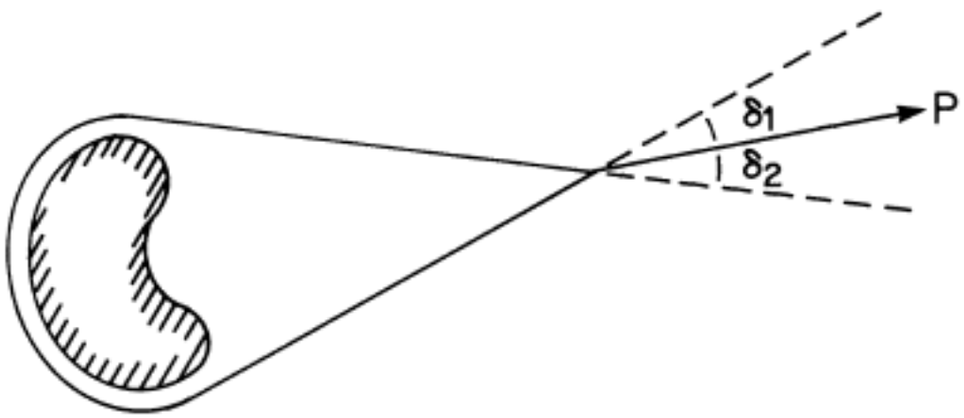
Stability of two rings



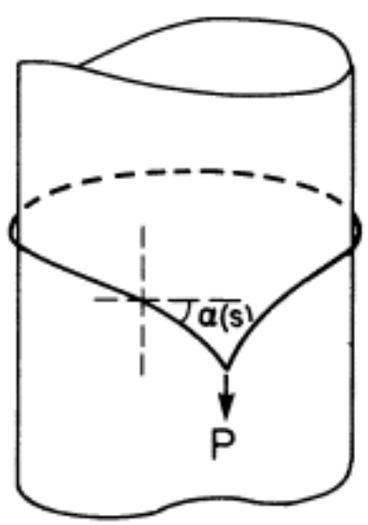
With Gaetano Napoli
Tom Mullin

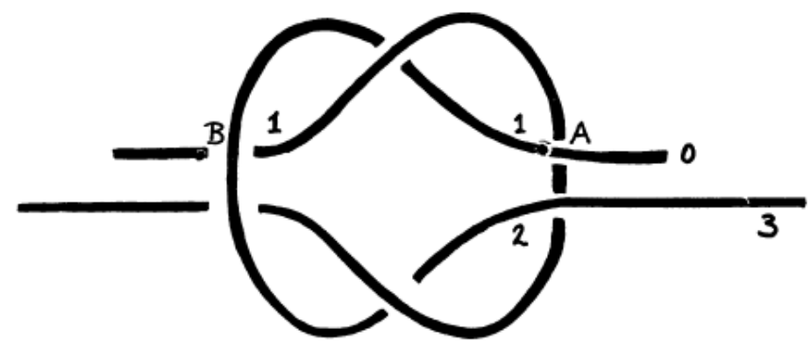
Contact problems

- Ropes in equilibrium (1987, with Maddocks)

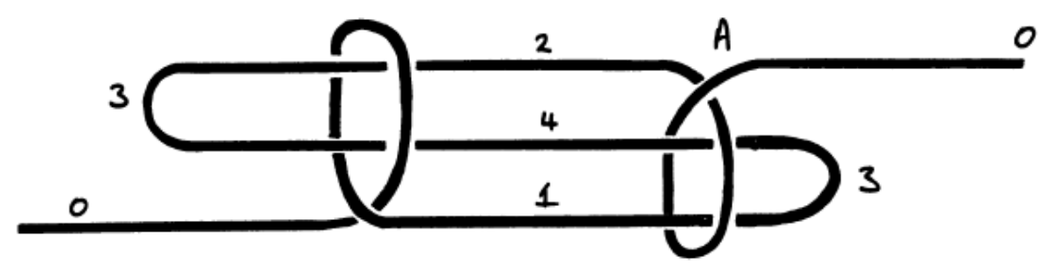


(a)





(a)

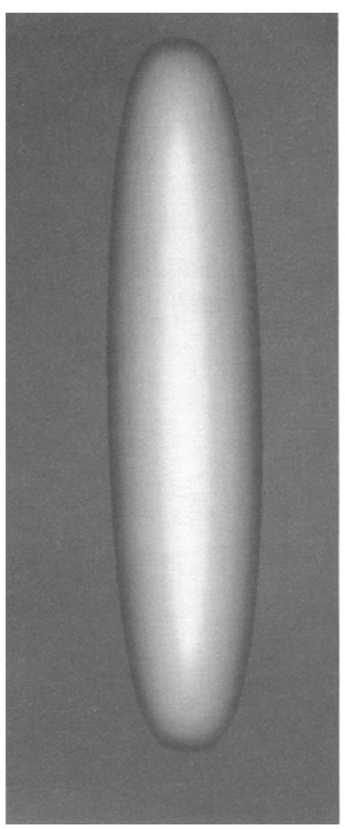


(b)

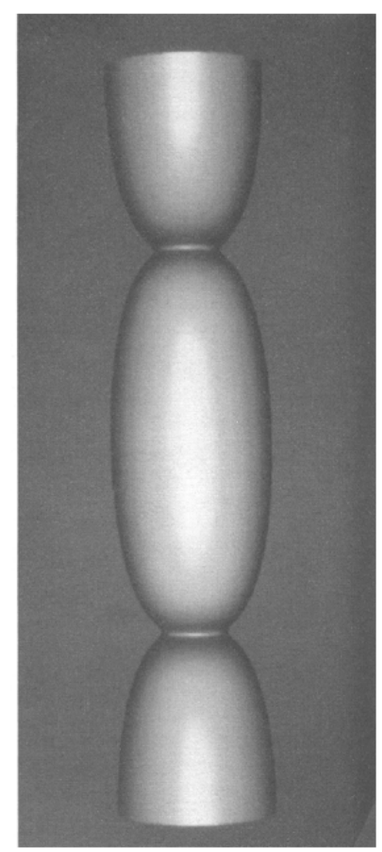
Optimal/Ideal shapes

- Strongest column (1960)
- Tallest column (1966, with Niordson)
- Rope of minimum elongation (1984, with Verma)

"To find the curve which by its revolution around an axis determines the column of greatest efficiency"



Clamped



Hinged

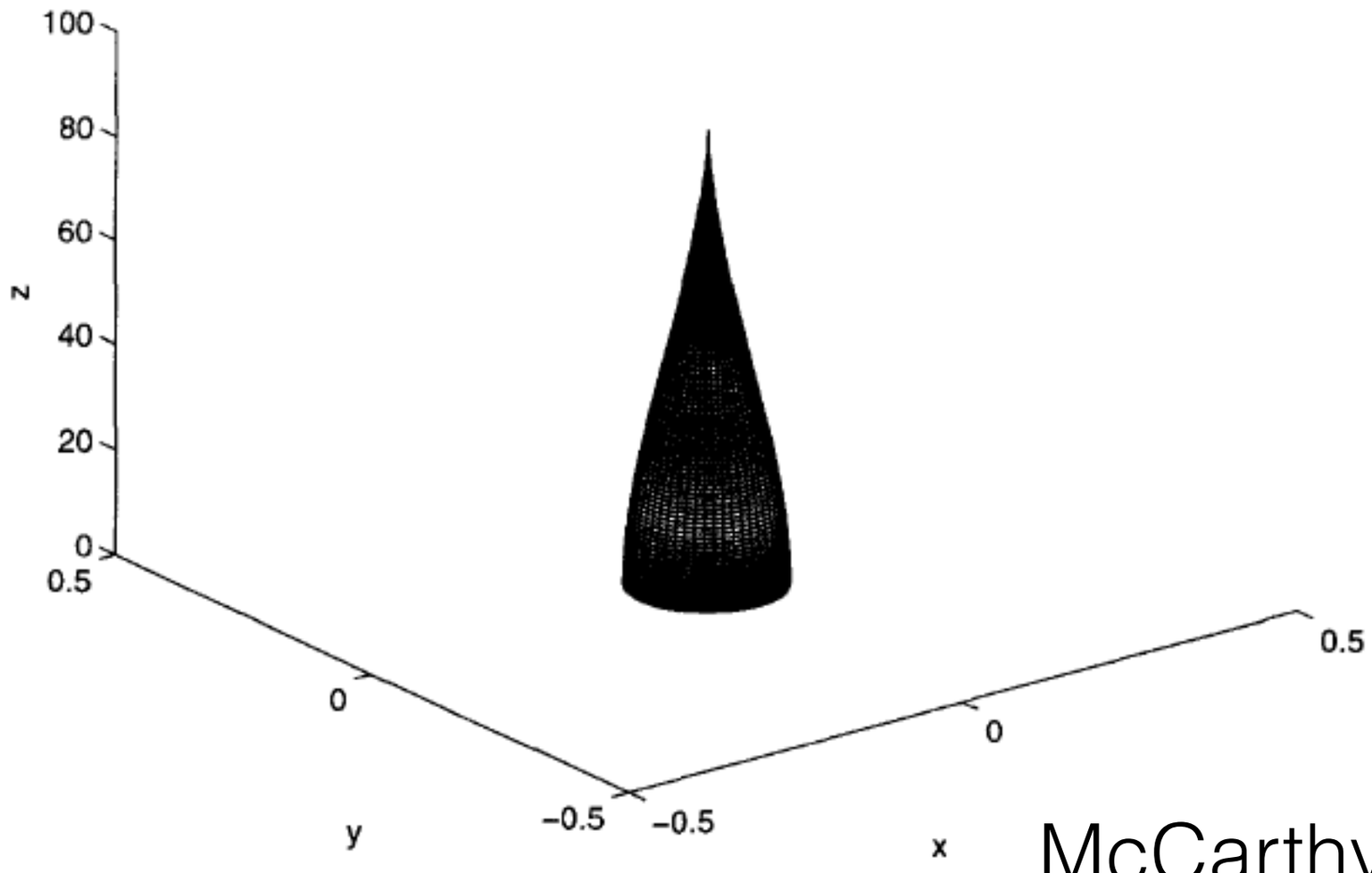
Cox 1992

Optimal/Ideal shapes

- Strongest column (1960)
- Tallest column (1966, with Niordson)
- Rope of minimum elongation (1984, with Verma)

Optimal/Ideal shapes

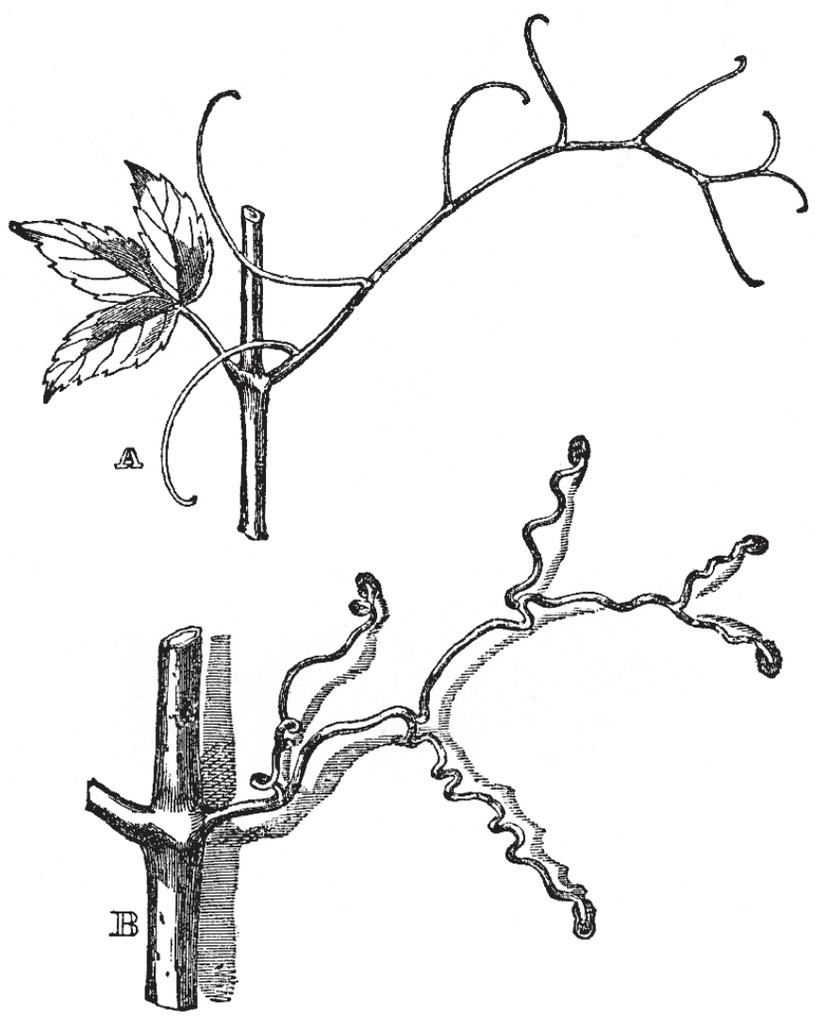
- Strongest column (1960)
- Tallest column (1966, with Niordson)
- Rope of minimum elongation (1984, with Verma)

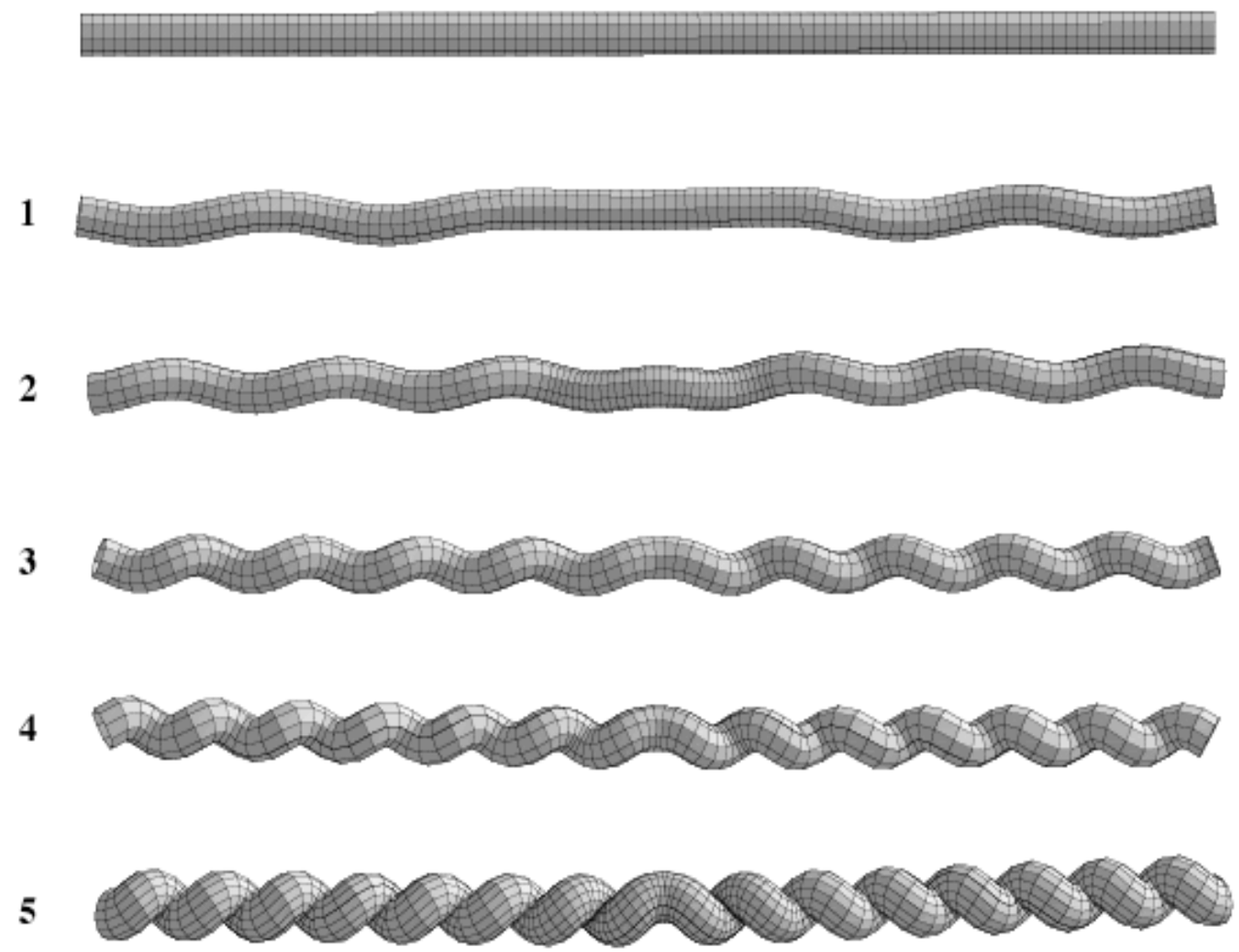


McCarthy, 1999

Optimal/Ideal shapes

- Tendril shape (1984)

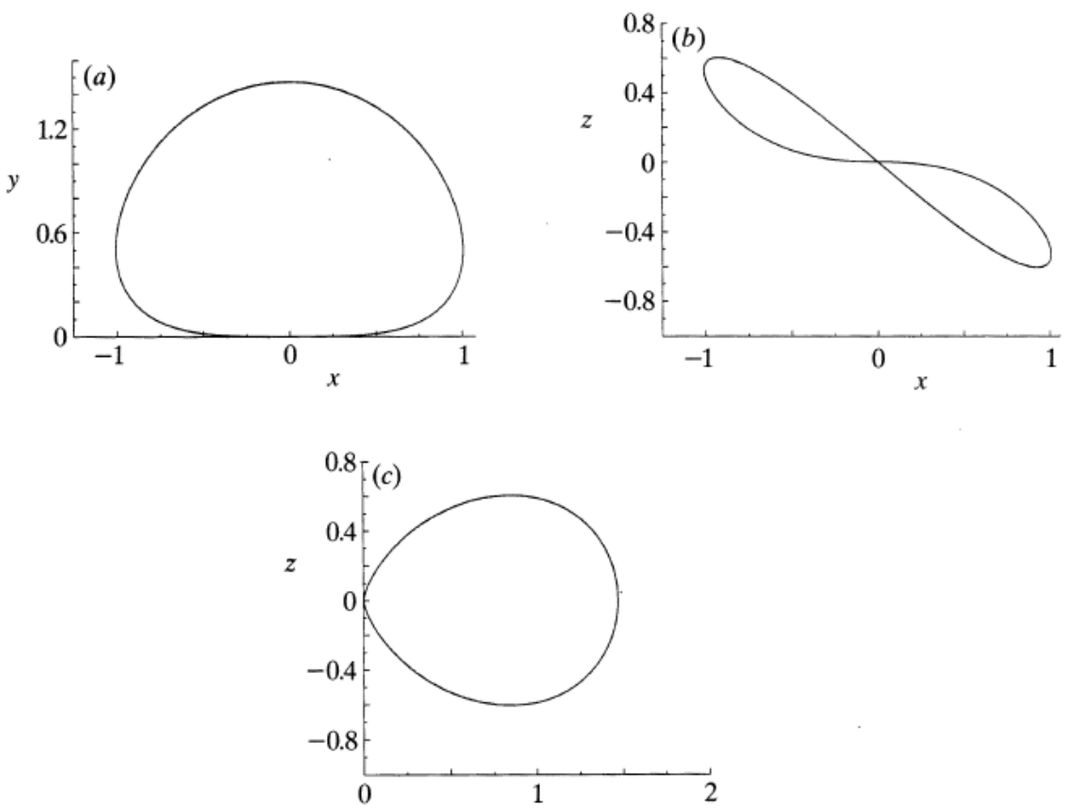


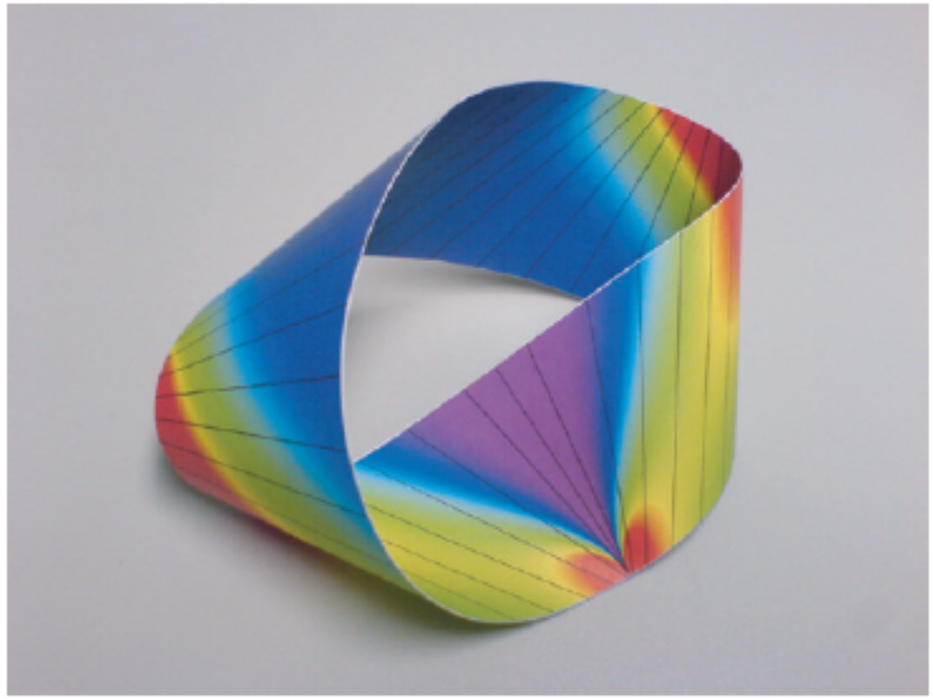


McMillen AG 2002

Optimal/Ideal shapes

- Mobius band (1993, with Mahadevan)
- Coiling of ropes (1995, with Mahadevan)

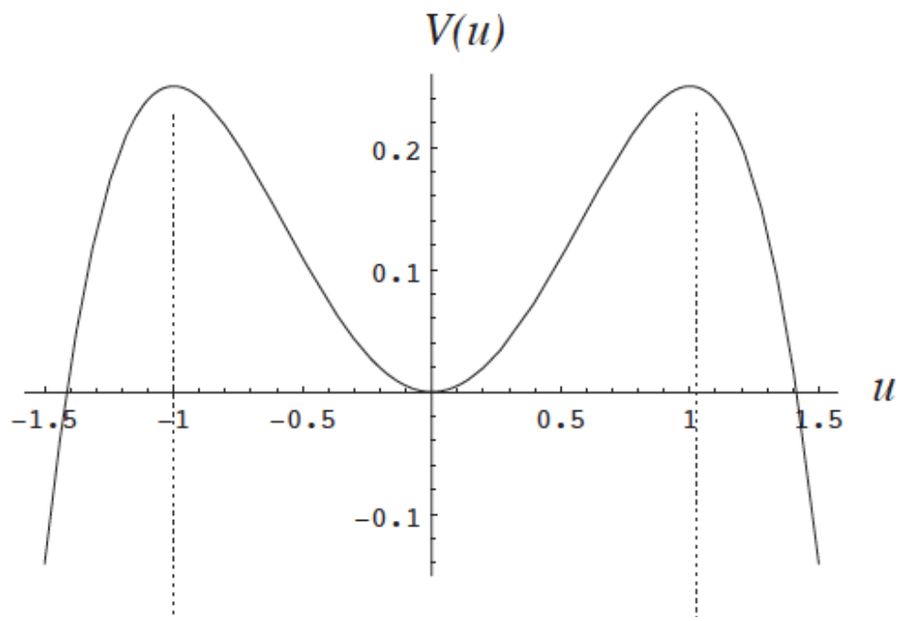




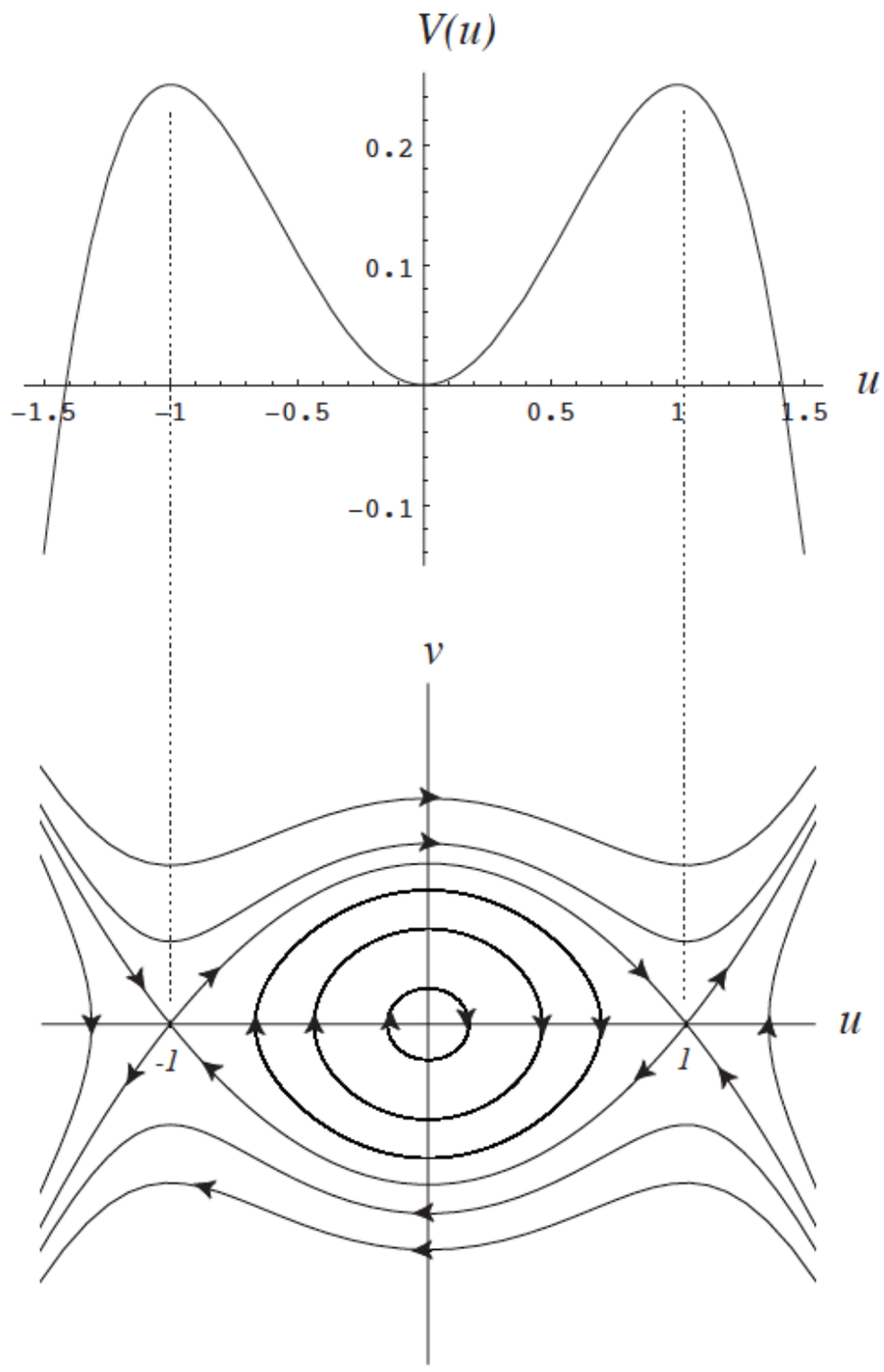
Starostin-van der Heijden, 2007

Stability methods for one-dimensional problems (work with Th. Lessinnes)

Dynamics in phase space

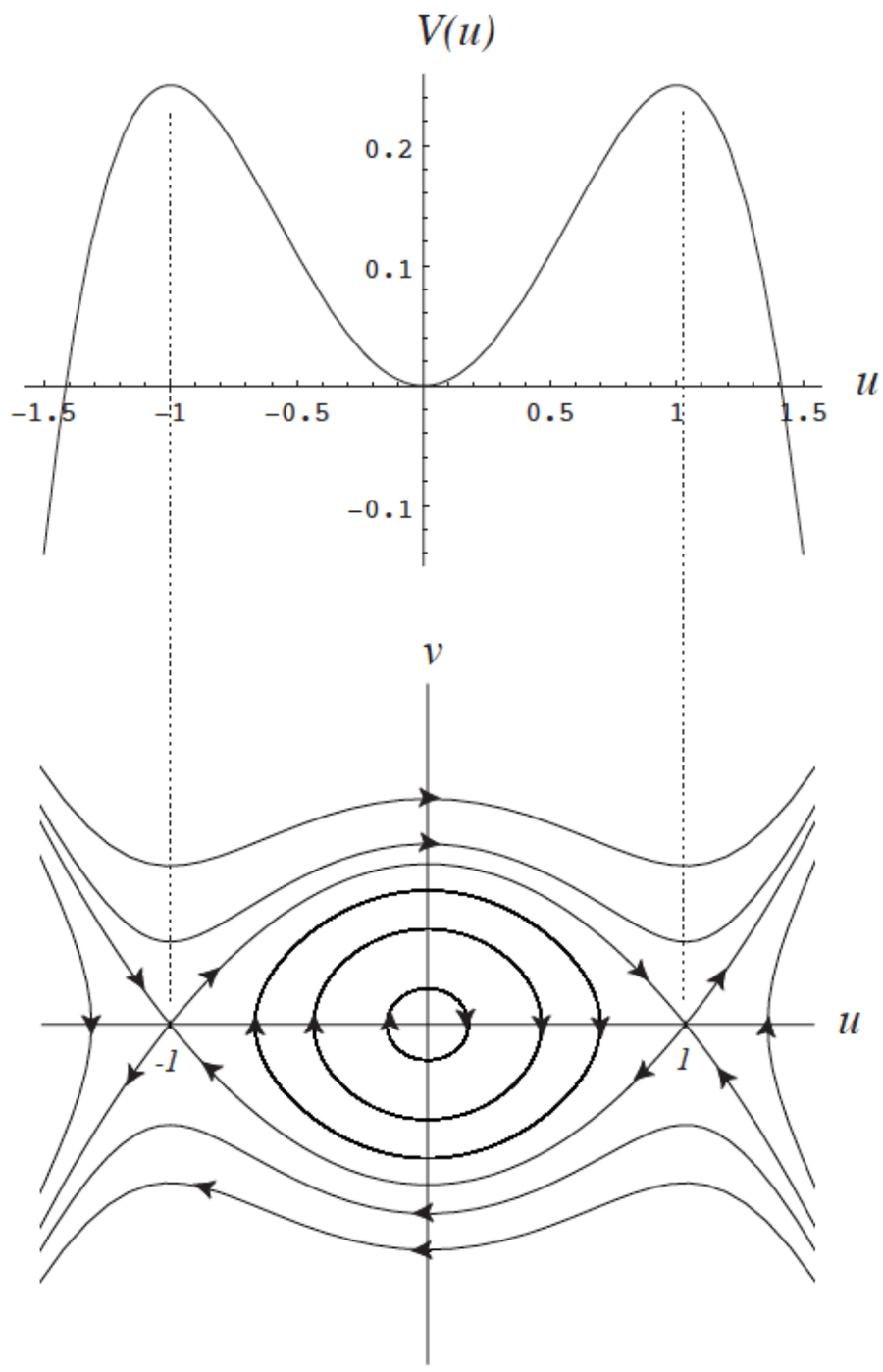


Dynamics in phase space

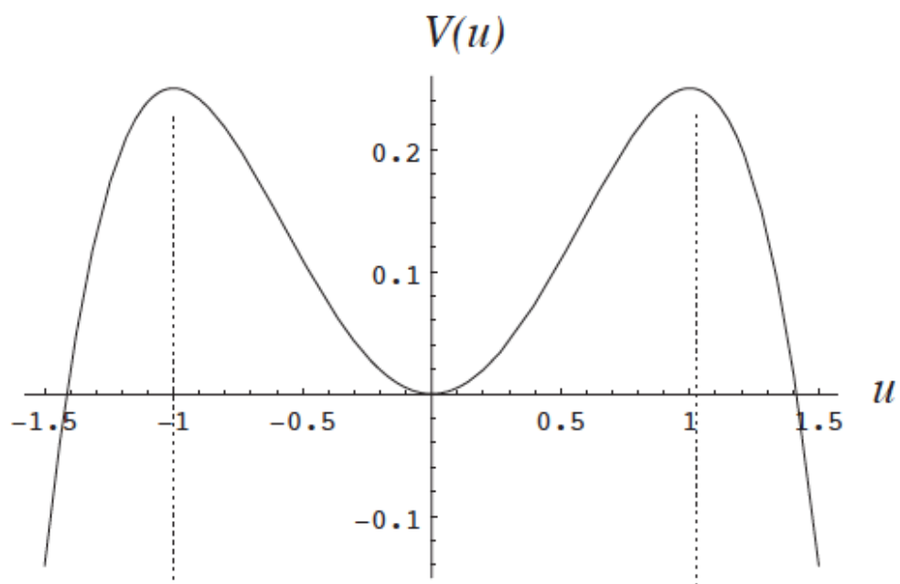


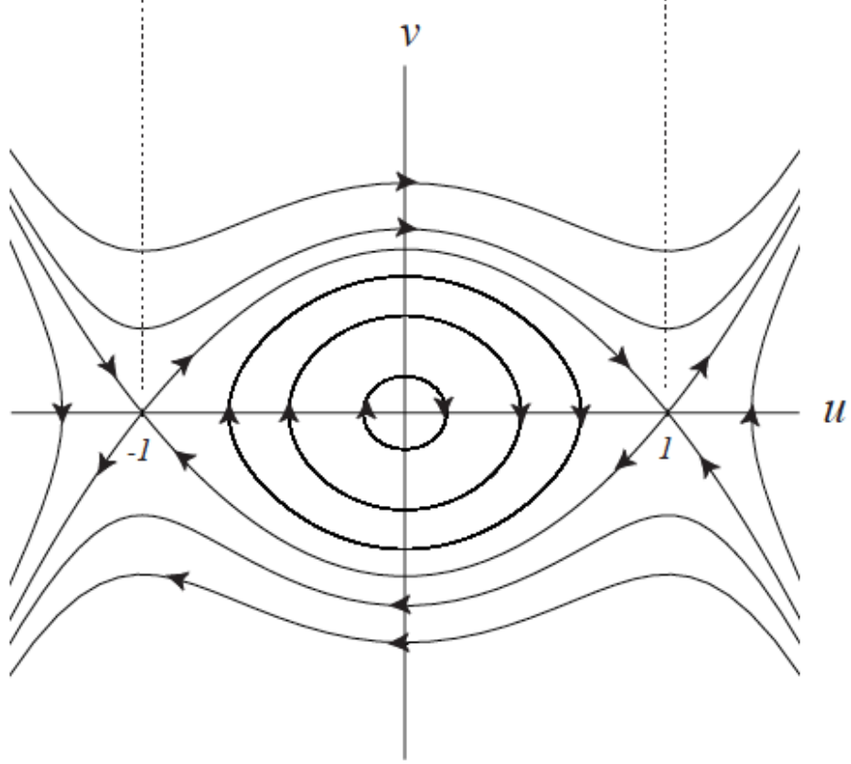
Dynamics in phase space

- More generally

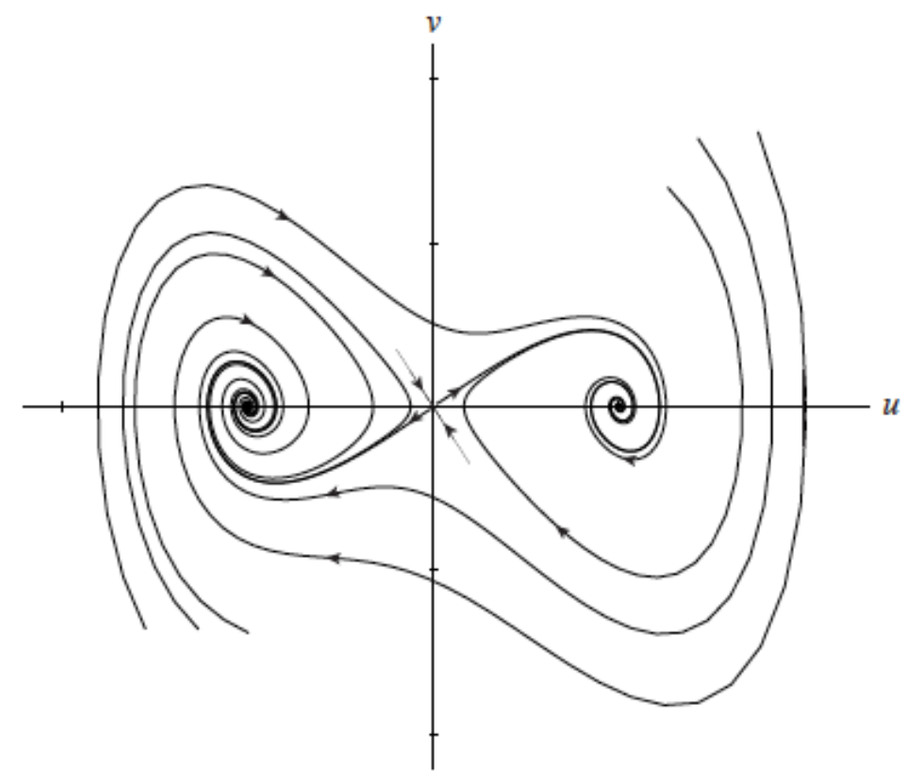


Dynamics in phase space

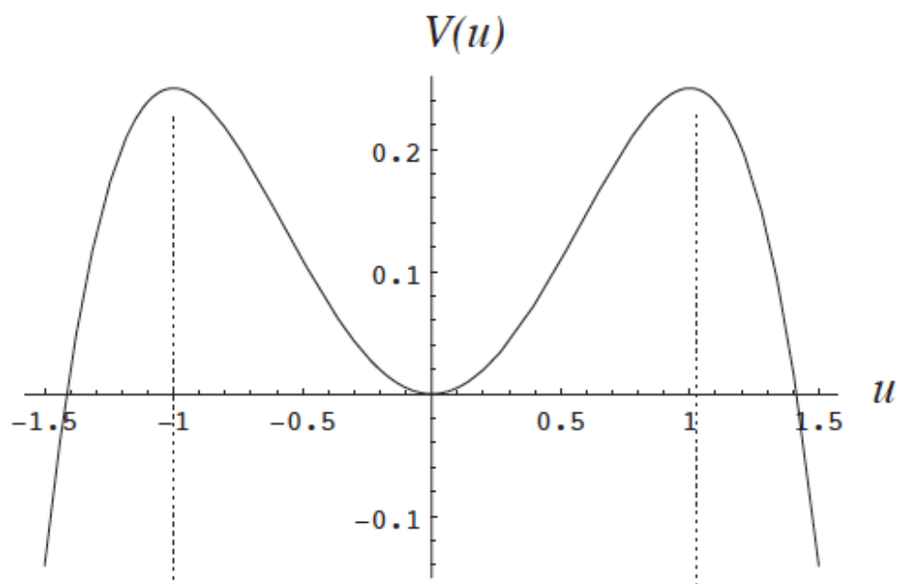


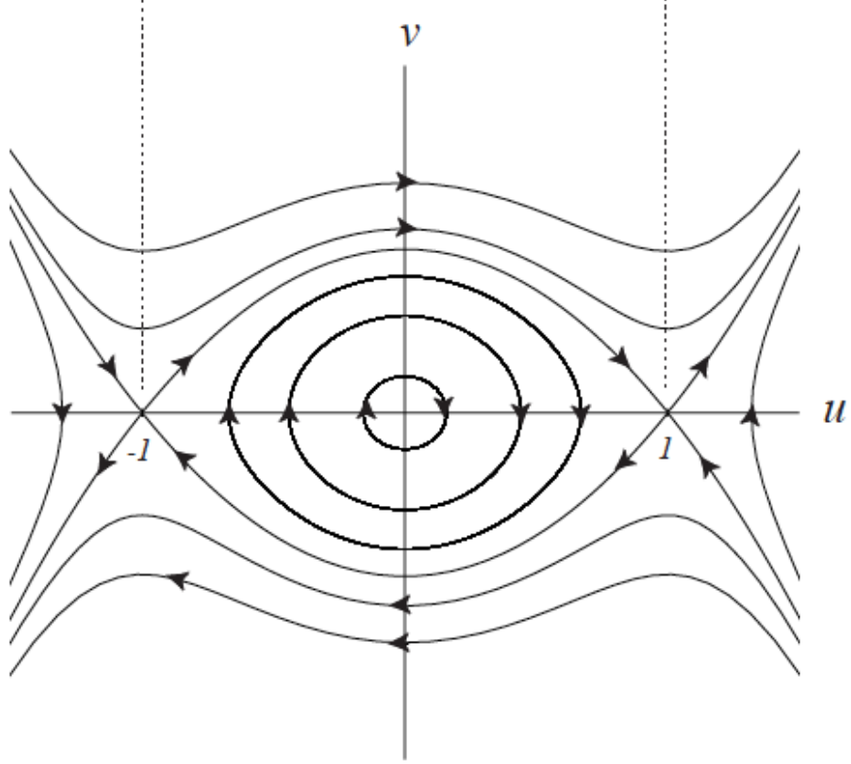


- More generally

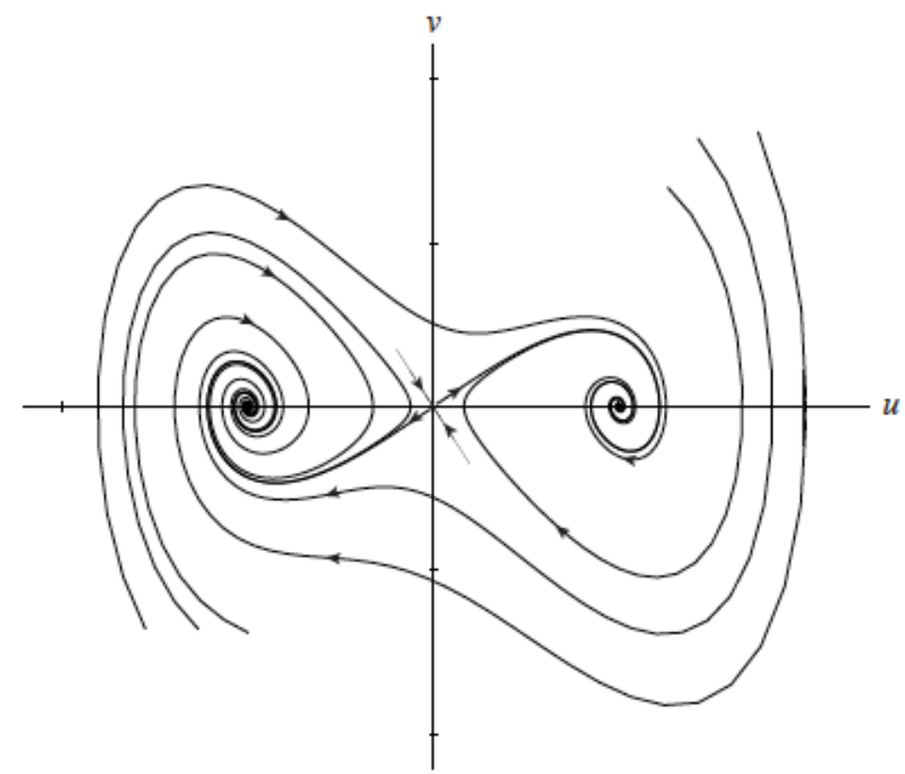


Dynamics in phase space



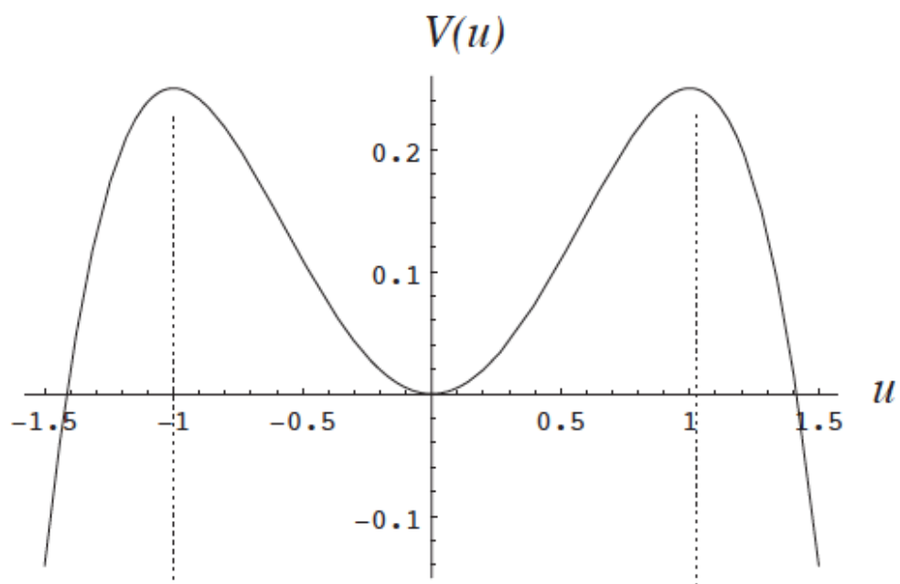


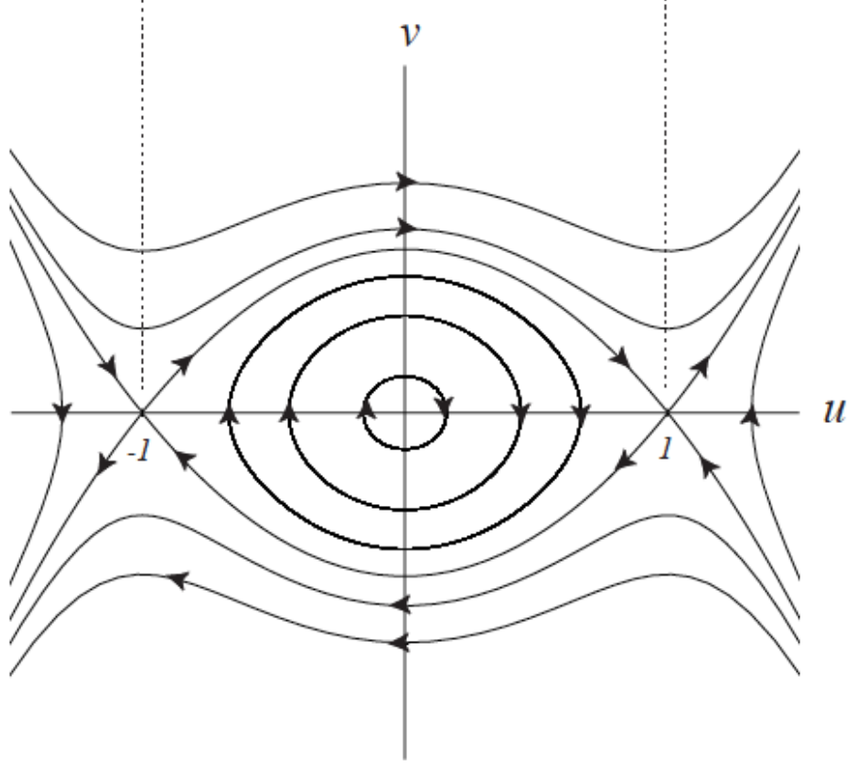
- More generally



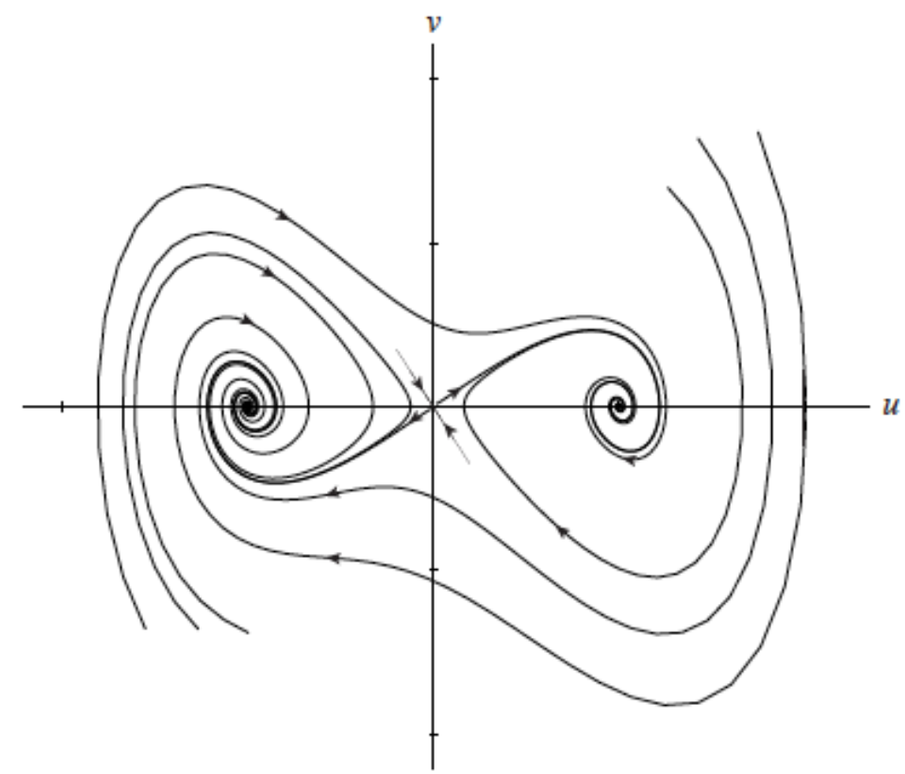
For initial value problems, the geometry of curves in phase space provides information on the stability of solutions.

Dynamics in phase space





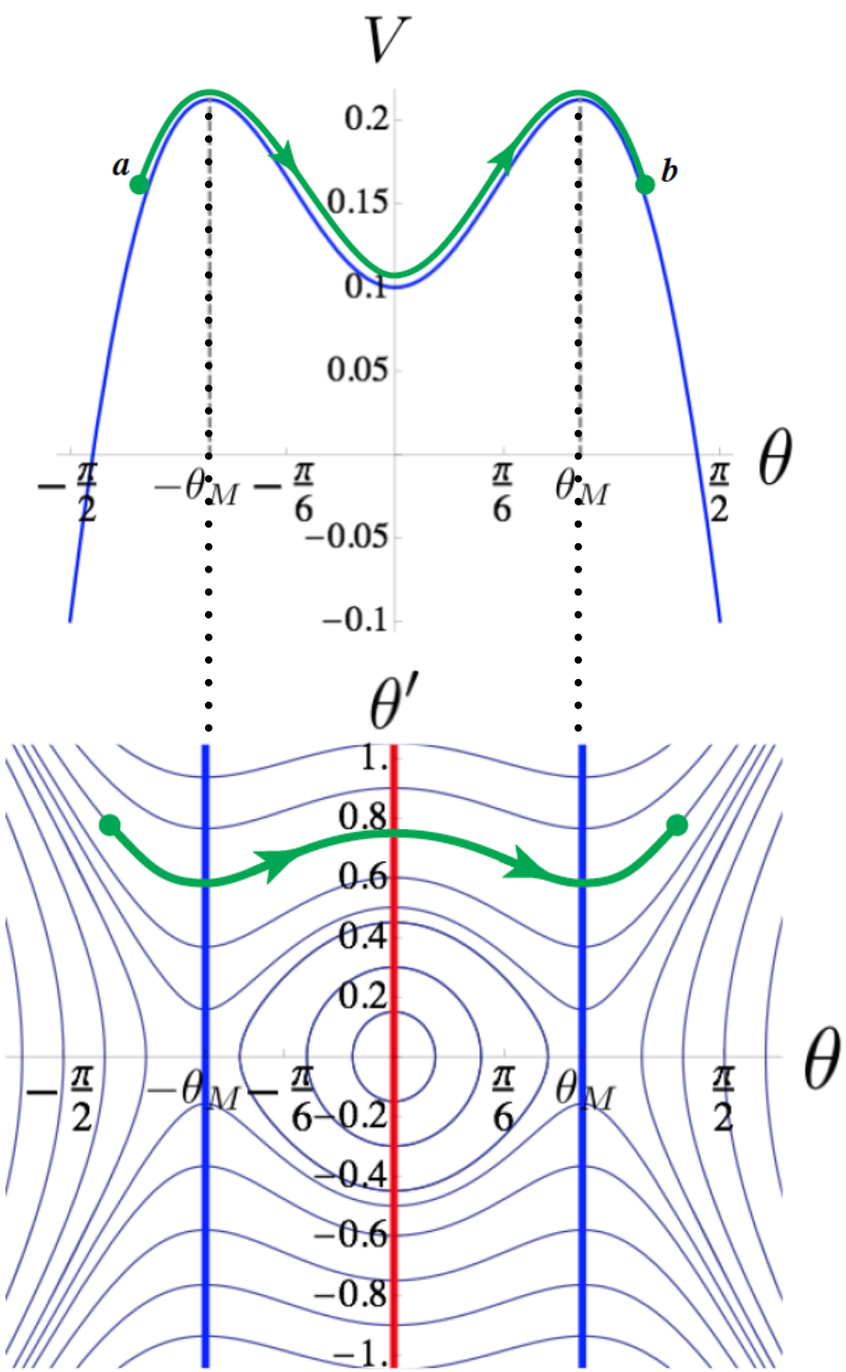
- More generally



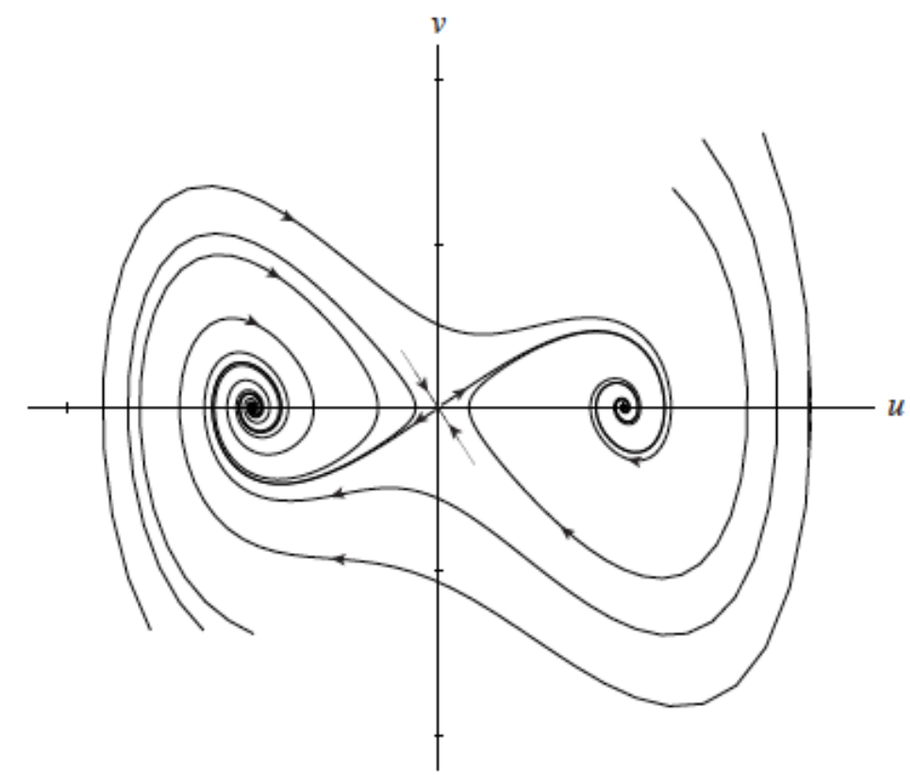
For initial value problems, the geometry of curves in phase space provides information on the stability of solutions.

For boundary value problems, equilibrium solutions also lie in phase space.

Dynamics in phase space



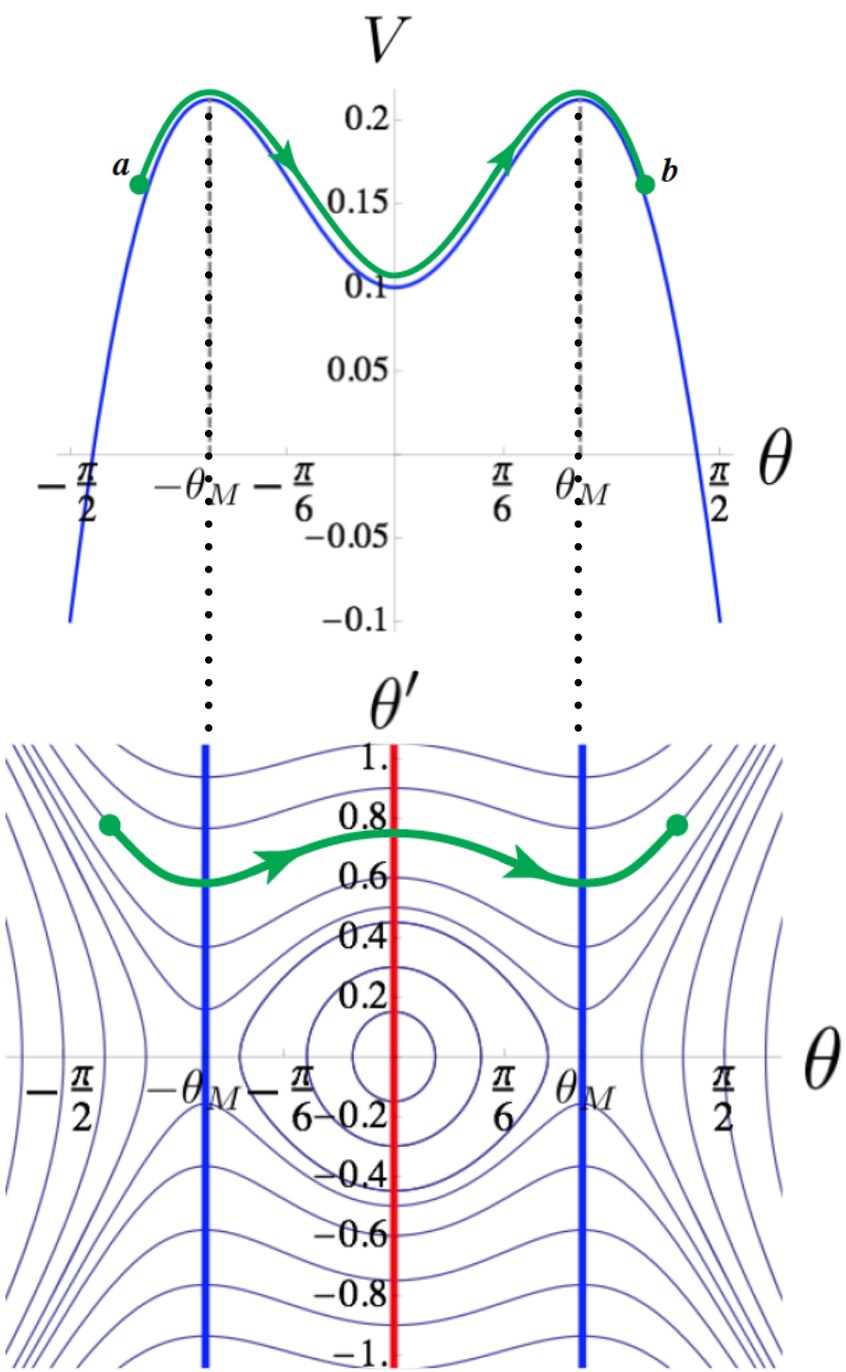
- More generally



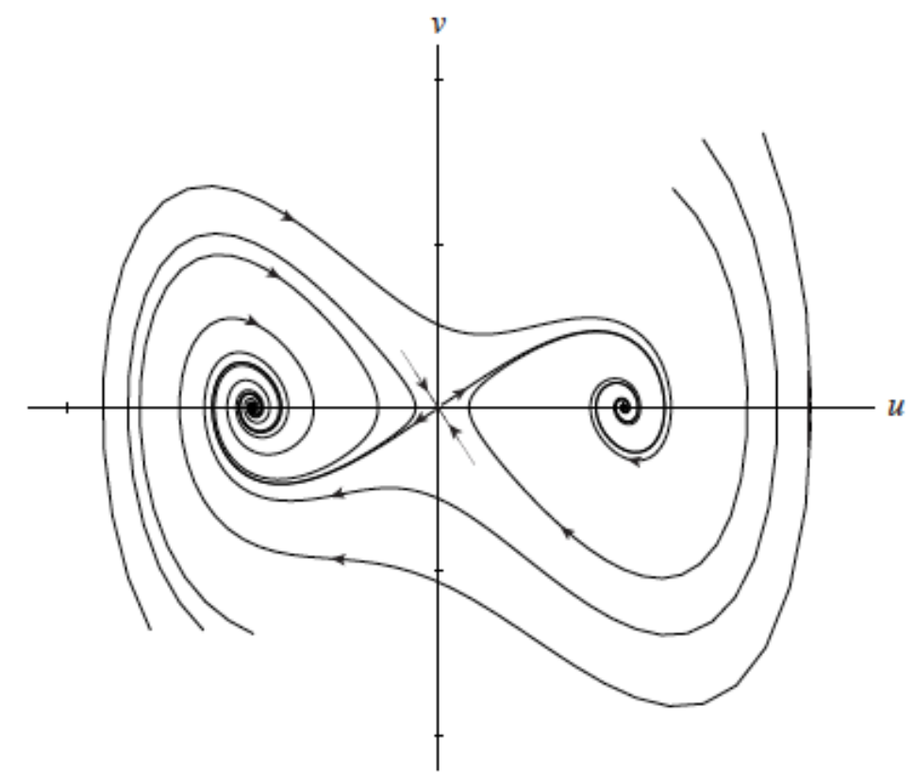
For initial value problems, the geometry of curves in phase space provides information on the stability of solutions.

For boundary value problems, equilibrium solutions also lie in phase space.

Dynamics in phase space



- More generally



For initial value problems, the geometry of curves in phase space provides information on the stability of solutions.

For boundary value problems, equilibrium solutions also lie in phase space.

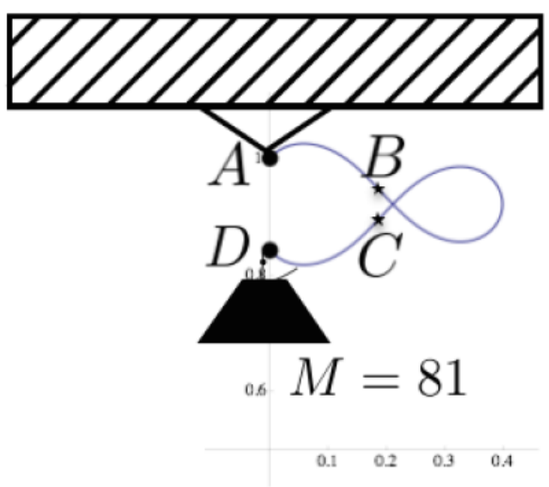
Stability?

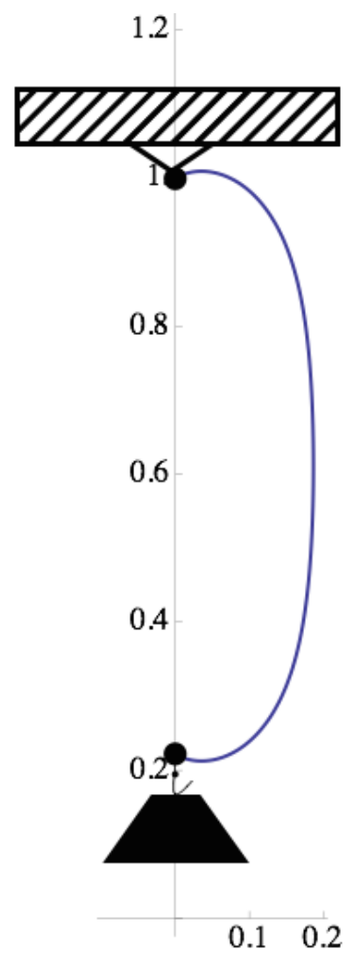
Example: hanging rod

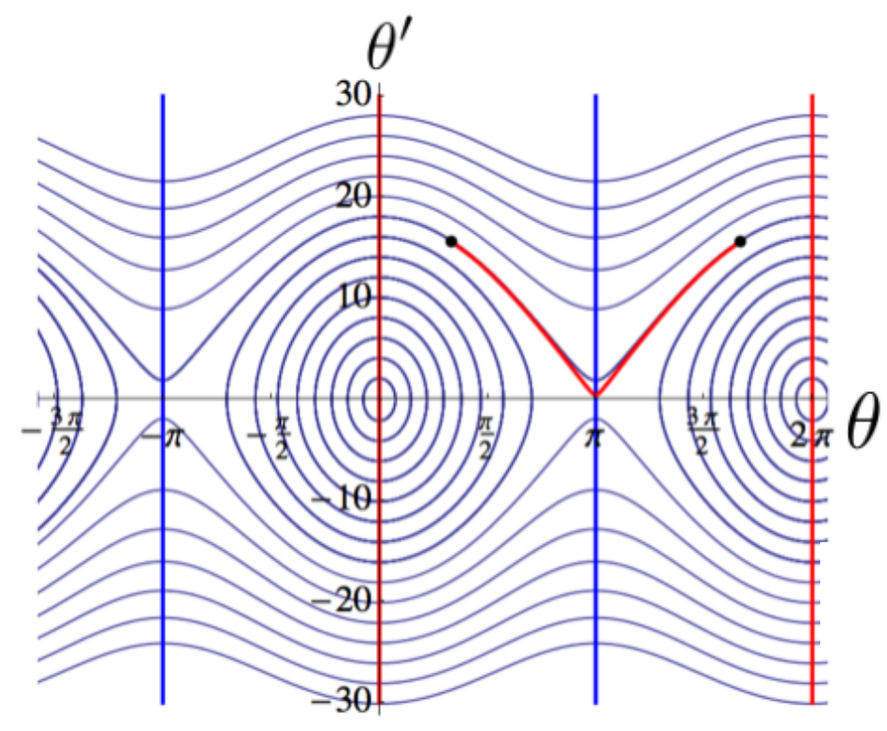
$$\mathscr{E} = \int_0^1 \frac{\left(\theta' - \sqrt{2Mv}\right)^2}{2} + M\cos\theta \mathrm{d}x, \qquad V = -M\cos\theta.$$

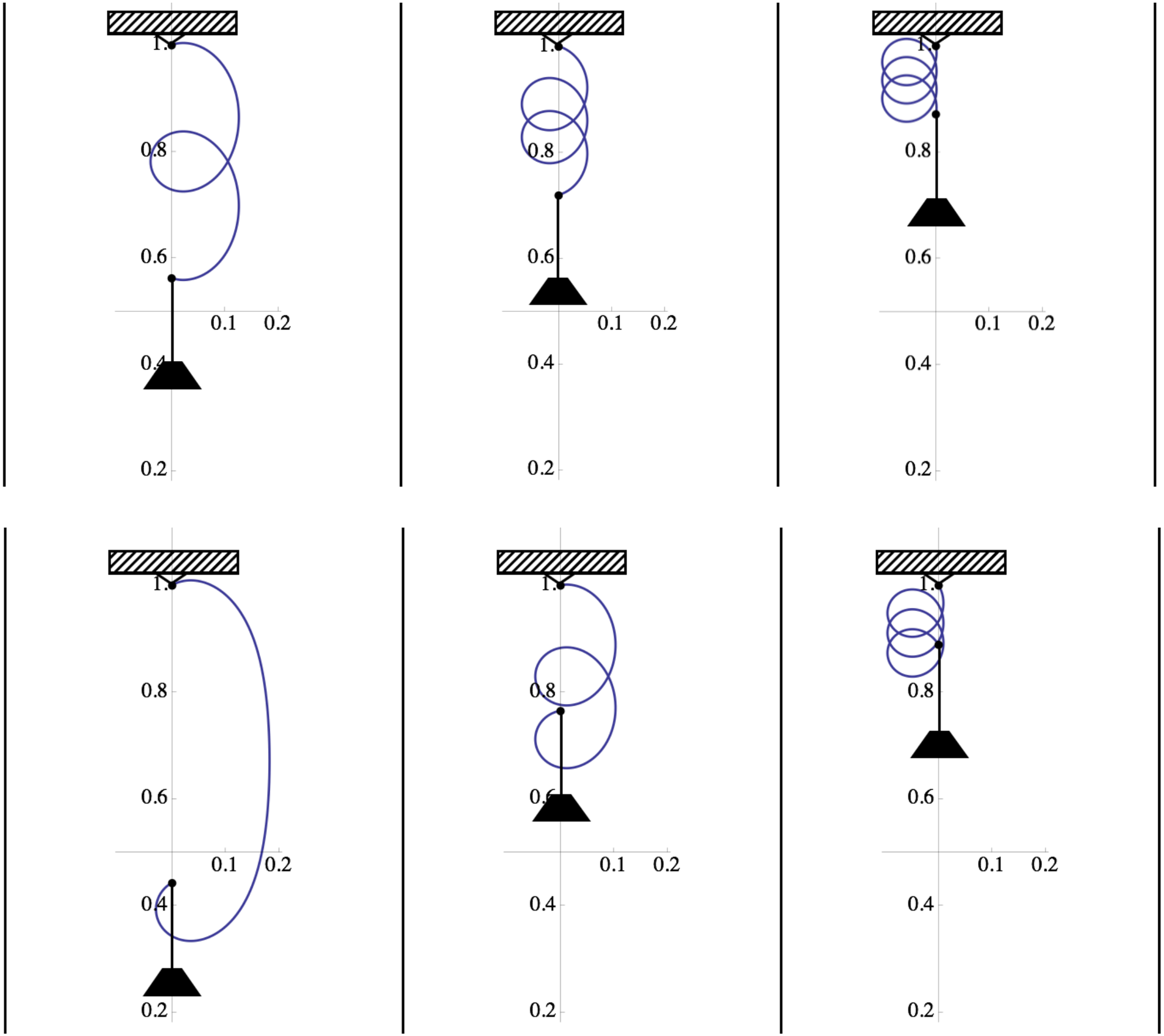
$$\boxed{\theta'' + M\sin\theta = 0, \qquad \theta'(0) = \theta'(1) = \sqrt{2Mv}.}$$

Take for instance:
v=1.5 and M=81








1.
0.8
0.6
0.4
0.2
0.1
0.2

Problem statement

Define the functional

$$\mathscr{E}[\theta] = \int_a^b \mathscr{L}\Big(\theta(s), \theta'(s)\Big)\ \mathrm{d}s$$

with

$$\mathscr{L}(\theta, \theta') = \frac{(\theta' - A)^2}{2} - V(\theta),$$

$$\theta(a) = \theta_a,\ \theta(b) = \theta_b$$

or

$$\theta'(a) = \theta'(b) = A.$$

Find a function θ which locally minimises $\mathscr{E}$: $\mathscr{E}[\theta + \epsilon\tau] > \mathscr{E}[\theta]$

Problem statement

Define the functional

$$\mathscr{E}[\theta] = \int_a^b \mathscr{L}\Big(\theta(s), \theta'(s)\Big)\ \mathrm{d}s$$

with

$$\mathscr{L}(\theta, \theta') = \frac{(\theta' - A)^2}{2} - V(\theta),$$

$$\theta(a) = \theta_a,\ \theta(b) = \theta_b$$

or

$$\theta'(a) = \theta'(b) = A.$$

Find a function θ which locally minimises $\mathscr{E}$: $\mathscr{E}[\theta + \epsilon\tau] > \mathscr{E}[\theta]$

To first order in ε,

$$\delta\mathscr{E}_\theta(\tau) = 0 \qquad \forall\tau \in \mathcal{C}'([a, b]) \qquad \Rightarrow \frac{\partial\mathscr{L}}{\partial\theta} - \frac{\mathrm{d}}{\mathrm{d}s}\frac{\partial\mathscr{L}}{\partial\theta'} = 0.$$

Problem statement

Define the functional

$$\mathscr{E}[\theta] = \int_a^b \mathscr{L}\Big(\theta(s), \theta'(s)\Big)\ \mathrm{d}s$$

with

$$\mathscr{L}(\theta, \theta') = \frac{(\theta' - A)^2}{2} - V(\theta),$$

$$\theta(a) = \theta_a,\ \theta(b) = \theta_b$$

or

$$\theta'(a) = \theta'(b) = A.$$

Find a function θ which locally minimises $\mathscr{E}$: $\mathscr{E}[\theta + \epsilon\tau] > \mathscr{E}[\theta]$

To first order in ε,

$$\delta\mathscr{E}_\theta(\tau) = 0 \qquad \forall\tau \in \mathcal{C}'([a,b]) \qquad \Rightarrow \frac{\partial\mathscr{L}}{\partial\theta} - \frac{\mathrm{d}}{\mathrm{d}s}\frac{\partial\mathscr{L}}{\partial\theta'} = 0.$$

To second order:

$$\delta^2\mathscr{E}_\theta(\tau) > 0 \qquad \forall\tau \in \mathcal{C}'([a,b])$$

Problem statement

Define the functional

$$\mathscr{E}[\theta] = \int_a^b \mathscr{L}\Big(\theta(s), \theta'(s)\Big)\ \mathrm{d}s$$

with

$$\mathscr{L}(\theta, \theta') = \frac{(\theta' - A)^2}{2} - V(\theta),$$

$$\theta(a) = \theta_a,\ \theta(b) = \theta_b$$

or

$$\theta'(a) = \theta'(b) = A.$$

Find a function θ which locally minimises $\mathscr{E}$: $\mathscr{E}[\theta + \epsilon\tau] > \mathscr{E}[\theta]$

To first order in ε,

$$\delta\mathscr{E}_\theta(\tau) = 0 \qquad \forall\tau \in \mathcal{C}'([a,b]) \qquad \Rightarrow \frac{\partial\mathscr{L}}{\partial\theta} - \frac{\mathrm{d}}{\mathrm{d}s}\frac{\partial\mathscr{L}}{\partial\theta'} = 0.$$

To second order:

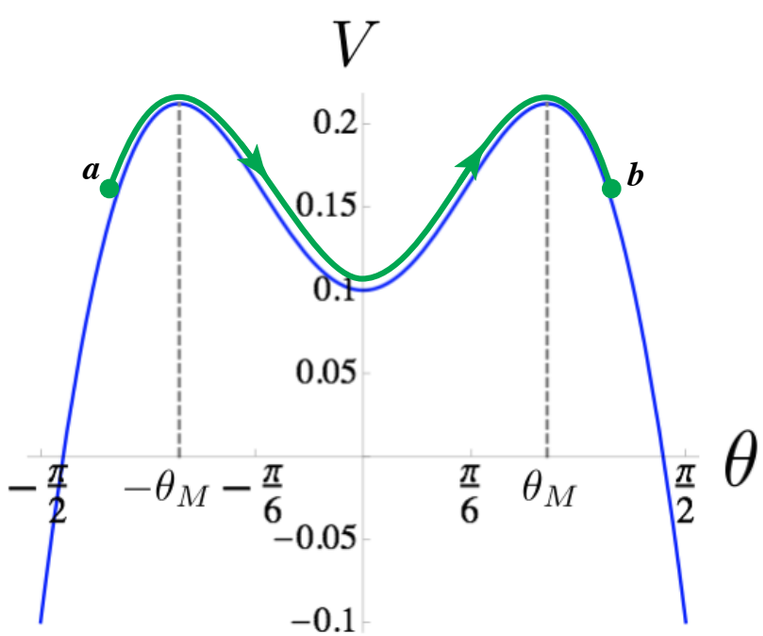
$$\delta^2\mathscr{E}_\theta(\tau) > 0 \qquad \forall\tau \in \mathcal{C}'([a,b])$$

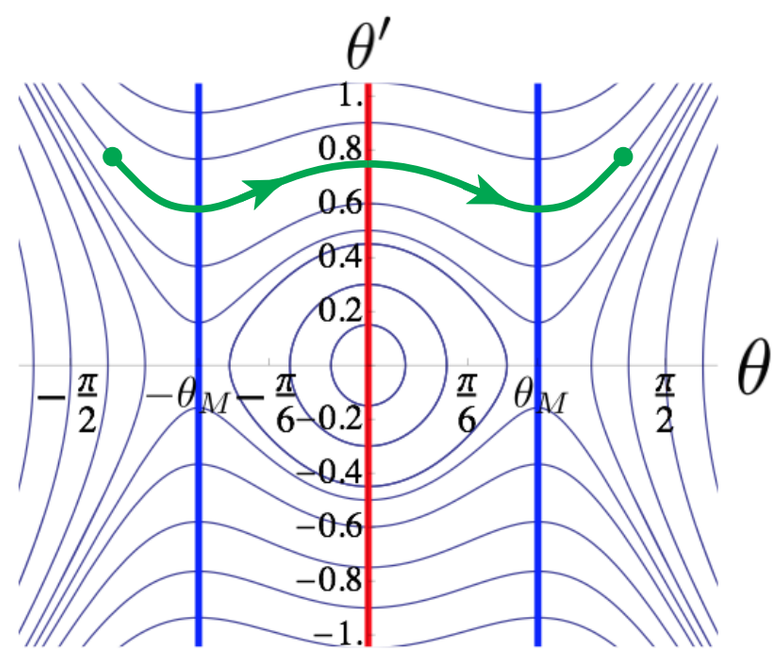
$$\mathcal{C}'([a,b]) \equiv \{f \in C^1([a,b]) \setminus \{0\} : f'(a) = f'(b) = 0\}$$

Theorem 2 [Lessinnes&AG, 2017]: Consider a stationary solution of

$$\mathscr{E}[\theta] = \int_a^b \frac{(\theta'(s) - A)^2}{2} - V\big(\theta(s)\big)\mathrm{d}s \qquad \theta'(a) = \theta'(b) = A.$$

and its corresponding solution in the phase plane $\gamma : s \in [a, b] \to \big(\theta(s), \theta'(s)\big)$



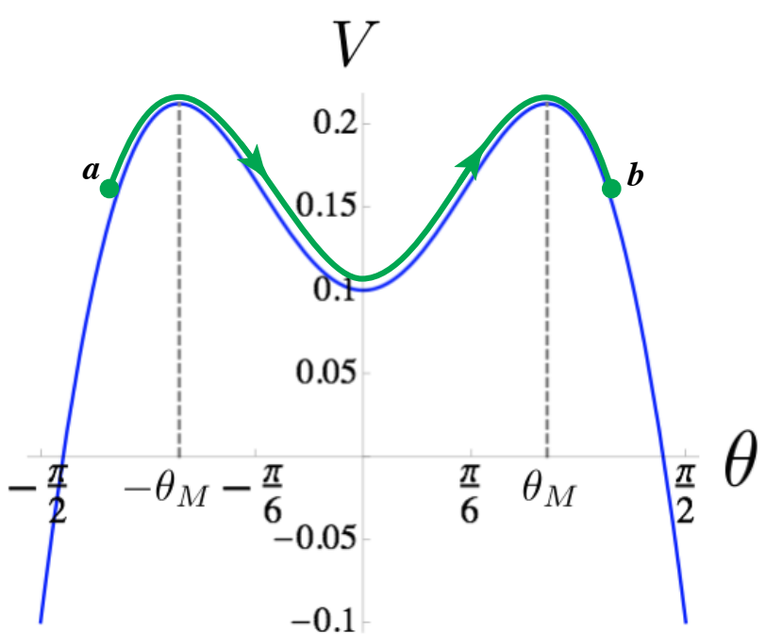


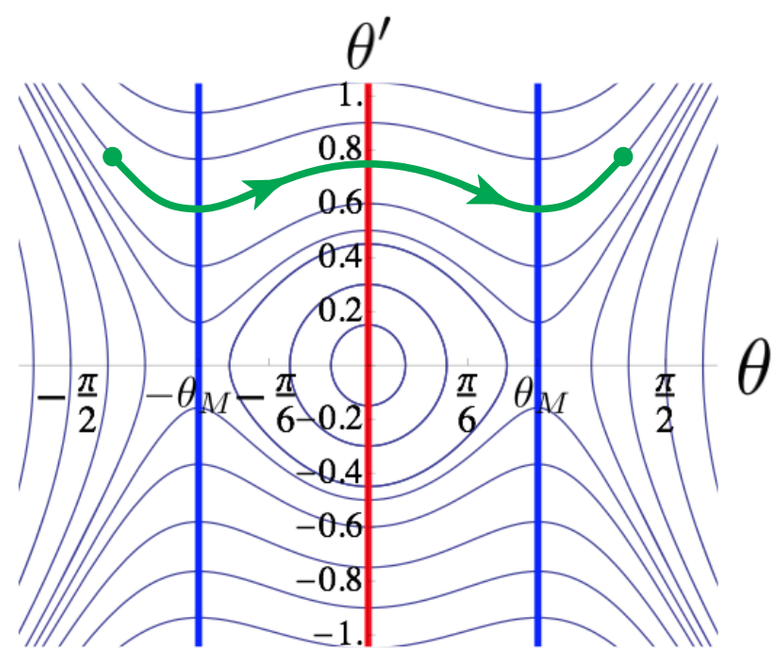
Theorem 2 [Lessinnes&AG, 2017]: Consider a stationary solution of

$$\mathscr{E}[\theta] = \int_a^b \frac{(\theta'(s) - A)^2}{2} - V\big(\theta(s)\big)\mathrm{d}s \qquad \theta'(a) = \theta'(b) = A.$$

and its corresponding solution in the phase plane $\gamma : s \in [a,b] \to \big(\theta(s), \theta'(s)\big)$

Define $\mathbf{I}$ = the number of max crossing minus the number of min crossing
= number of summits - number of valleys
= number of blue lines crossed - number of red lines crossed





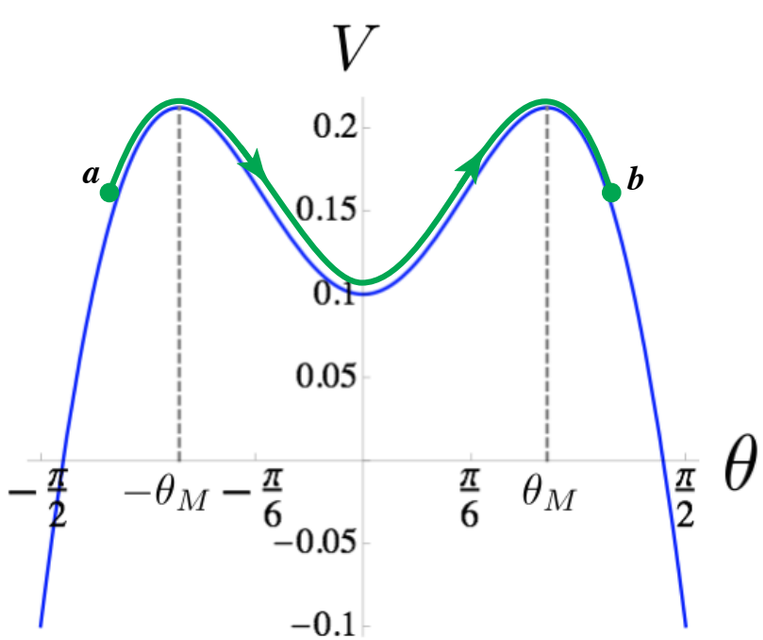
Theorem 2 [Lessinnes&AG, 2017]: Consider a stationary solution of

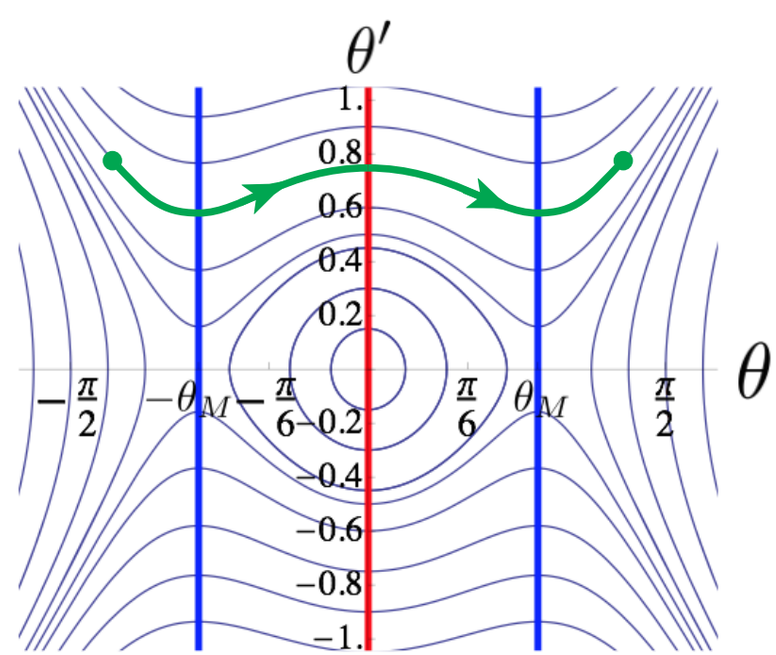
$$\mathscr{E}[\theta] = \int_a^b \frac{(\theta'(s) - A)^2}{2} - V\big(\theta(s)\big)\mathrm{d}s \qquad \theta'(a) = \theta'(b) = A.$$

and its corresponding solution in the phase plane $\gamma : s \in [a,b] \to \big(\theta(s), \theta'(s)\big)$

Define $\mathbf{I}$ = the number of max crossing minus the number of min crossing
= number of summits - number of valleys
= number of blue lines crossed - number of red lines crossed

Then, if $\mathbf{I}>0$, the second variation is strictly positive (local minimum=stable)
if $\mathbf{I}<0$, the second variation is not positive (not a minimum=unstable)





Theorem 2 [Lessinnes&AG, 2017]: Consider a stationary solution of

$$\mathscr{E}[\theta] = \int_a^b \frac{(\theta'(s) - A)^2}{2} - V\big(\theta(s)\big)\mathrm{d}s \qquad \theta'(a) = \theta'(b) = A.$$

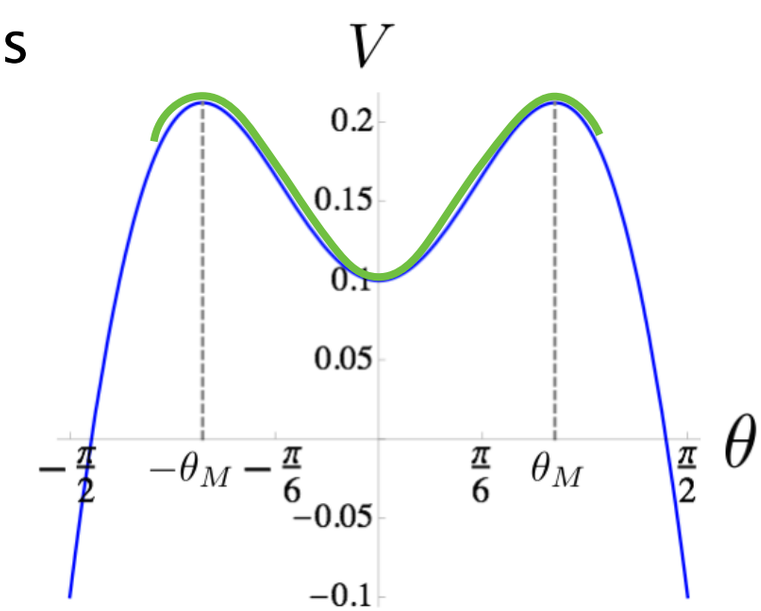
and its corresponding solution in the phase plane $\gamma : s \in [a,b] \to \big(\theta(s), \theta'(s)\big)$

Define $\mathbf{I}$ = the number of max crossing minus the number of min crossing
= number of summits - number of valleys
= number of blue lines crossed - number of red lines crossed

Then, if $\mathbf{I}>0$, the second variation is strictly positive (local minimum=stable)
if $\mathbf{I}<0$, the second variation is not positive (not a minimum=unstable)

If $\mathbf{I}=0$, Define $\mathbf{J}$ as the weighted difference in slopes at the end points

$$\mathbf{J} = A(\mathbf{V}'(\theta(b)) - \mathbf{V}'(\theta(a)))$$



Theorem 2 [Lessinnes&AG, 2017]: Consider a stationary solution of

$$\mathscr{E}[\theta] = \int_a^b \frac{(\theta'(s) - A)^2}{2} - V\big(\theta(s)\big)\mathrm{d}s \qquad \theta'(a) = \theta'(b) = A.$$

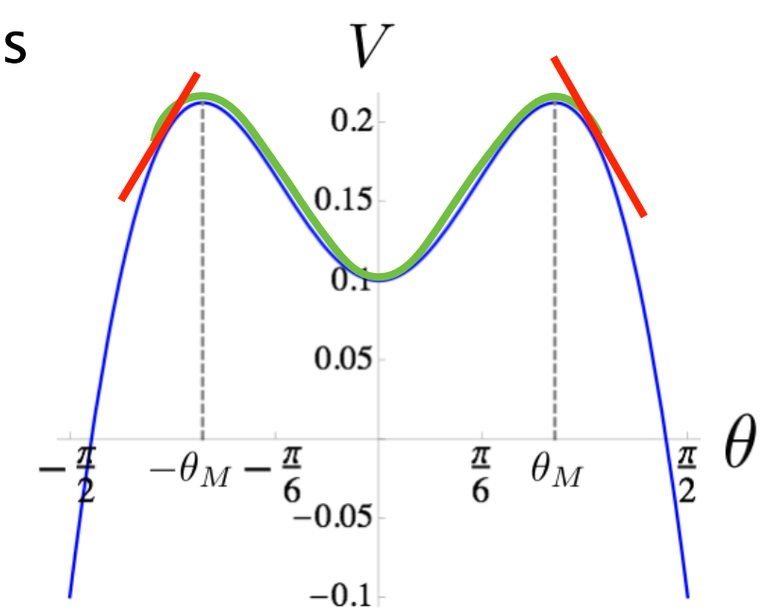
and its corresponding solution in the phase plane $\gamma : s \in [a,b] \to \big(\theta(s), \theta'(s)\big)$

Define $\mathbf{I}$ = the number of max crossing minus the number of min crossing
= number of summits - number of valleys
= number of blue lines crossed - number of red lines crossed

Then, if $\mathbf{I}>0$, the second variation is strictly positive (local minimum=stable)
if $\mathbf{I}<0$, the second variation is not positive (not a minimum=unstable)

If $\mathbf{I}=0$, Define $\mathbf{J}$ as the weighted difference in slopes at the end points

$$\mathbf{J} = A(\mathbf{V}'(\theta(b)) - \mathbf{V}'(\theta(a)))$$



Theorem 2 [Lessinnes&AG, 2017]: Consider a stationary solution of

$$\mathscr{E}[\theta] = \int_a^b \frac{(\theta'(s) - A)^2}{2} - V\big(\theta(s)\big)\mathrm{d}s \qquad \theta'(a) = \theta'(b) = A.$$

and its corresponding solution in the phase plane $\gamma : s \in [a,b] \to \big(\theta(s), \theta'(s)\big)$

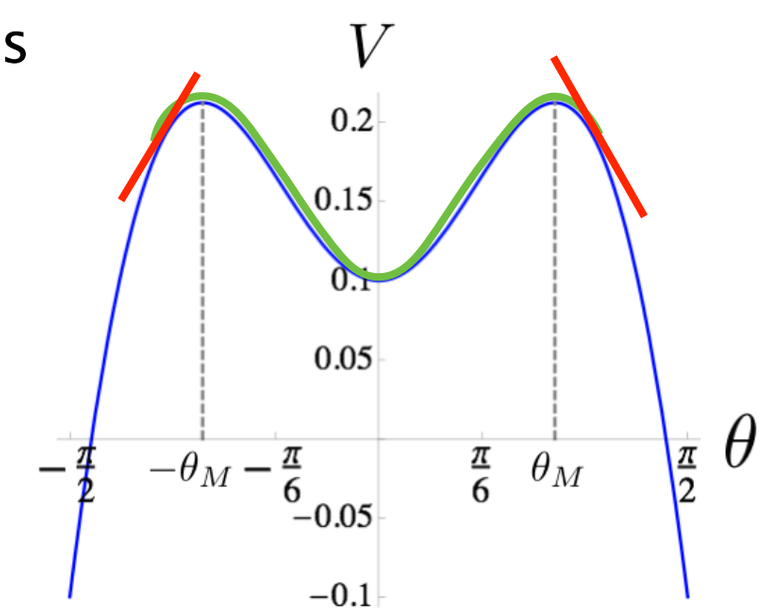
Define $\mathbf{I}$ = the number of max crossing minus the number of min crossing
= number of summits - number of valleys
= number of blue lines crossed - number of red lines crossed

Then, if $\mathbf{I}>0$, the second variation is strictly positive (local minimum=stable)
if $\mathbf{I}<0$, the second variation is not positive (not a minimum=unstable)

If $\mathbf{I}=0$, Define $\mathbf{J}$ as the weighted difference in slopes at the end points

$$\mathbf{J} = A(\mathbf{V}'(\theta(b)) - \mathbf{V}'(\theta(a)))$$

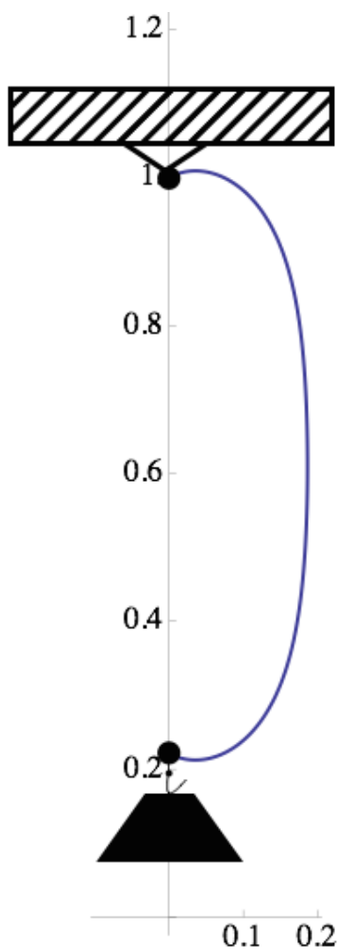
Then, if $\mathbf{J} \leqslant 0$, unstable
if $\mathbf{J}>0$, ? numerical

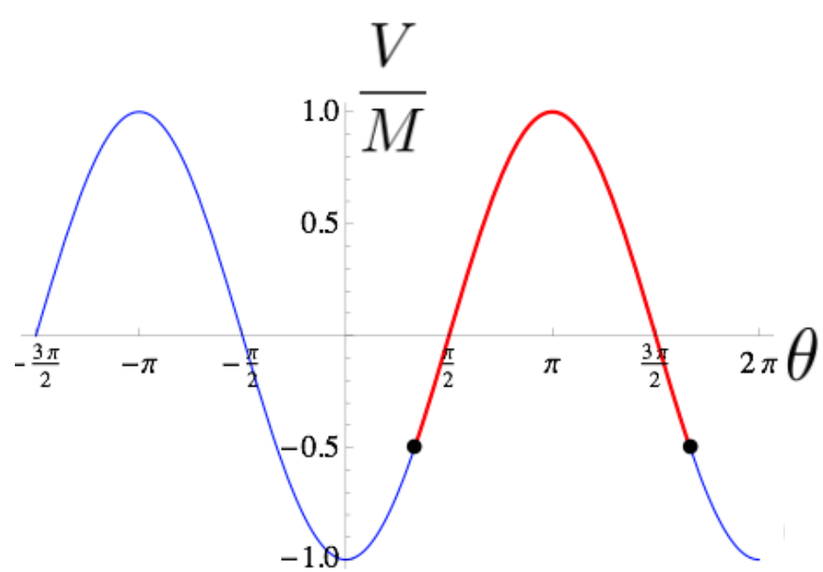


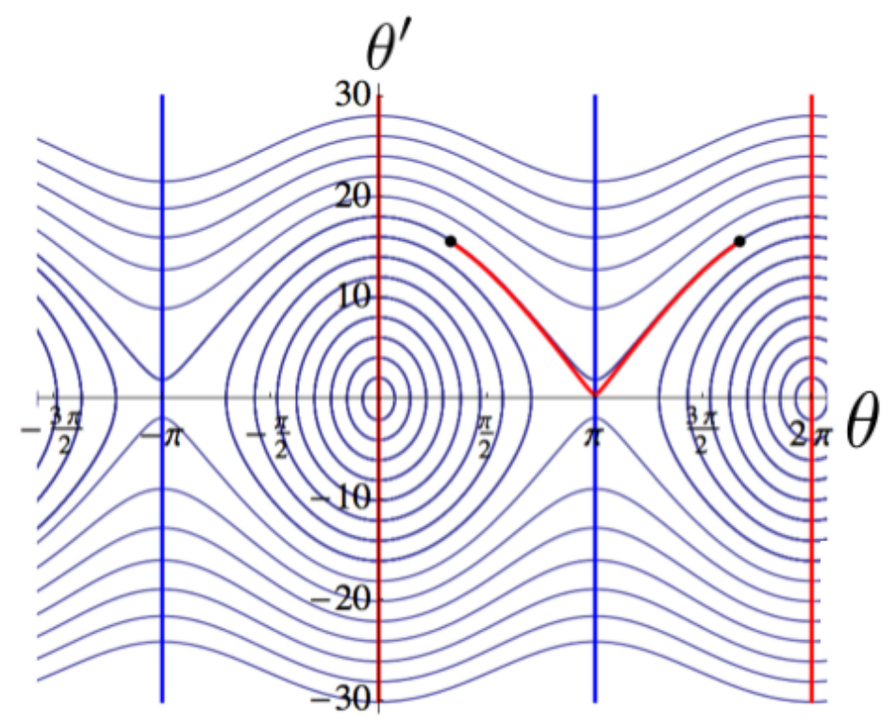
Example: hanging rod

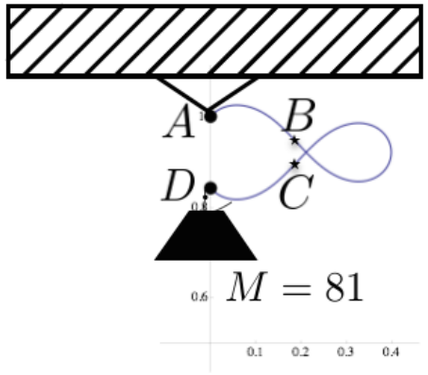
Stability: Count the number of max-number of min

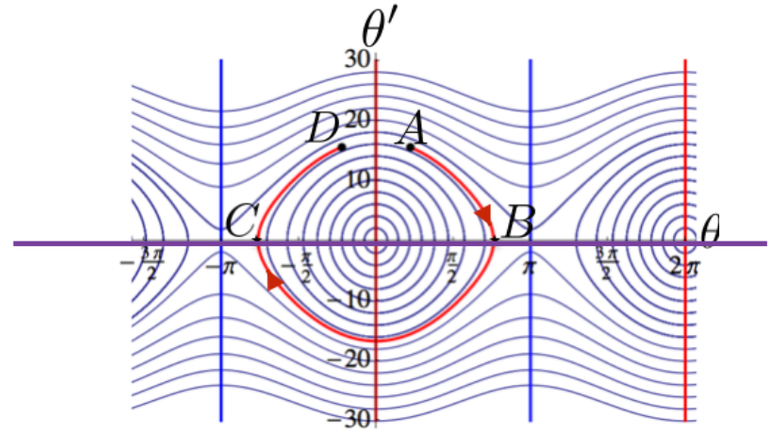
One max, no min $\Rightarrow \mathbf{I}>0 \Rightarrow$ Stable





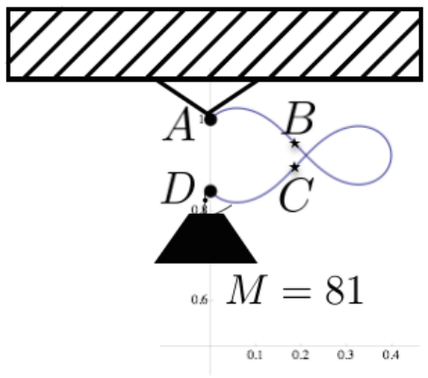


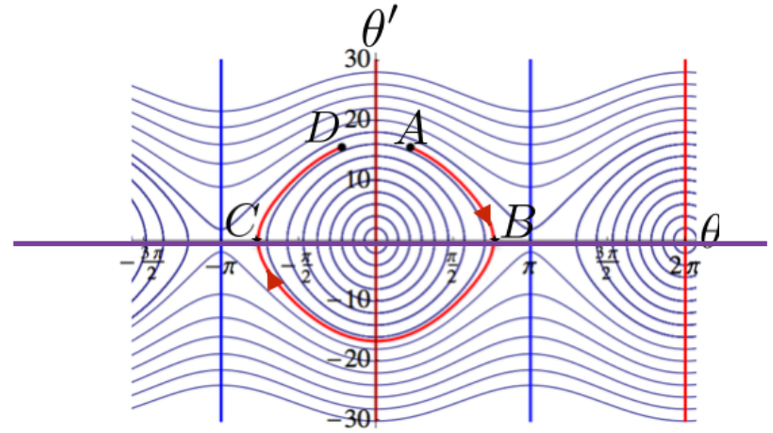




I=-1
Unstable

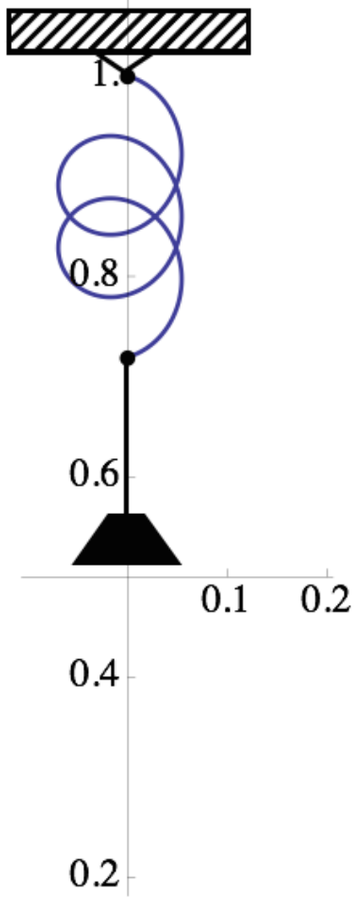
K=2
Dirichlet
unstable

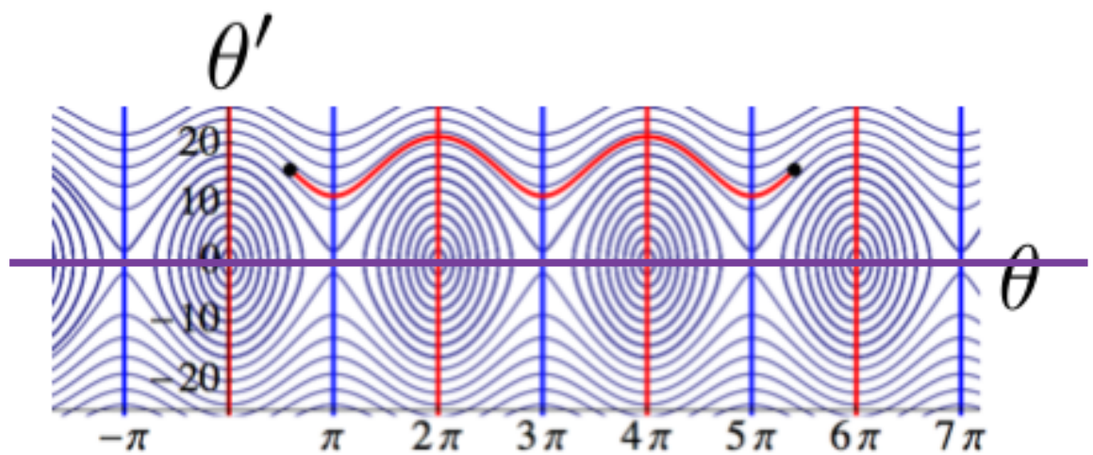




I=-1
Unstable

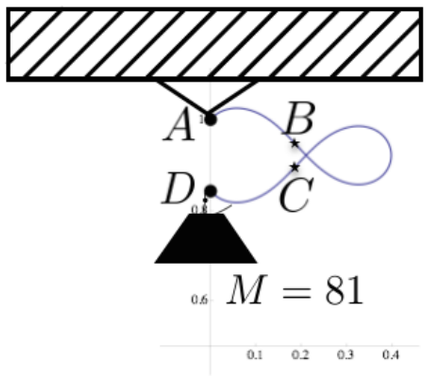
K=2
Dirichlet
unstable

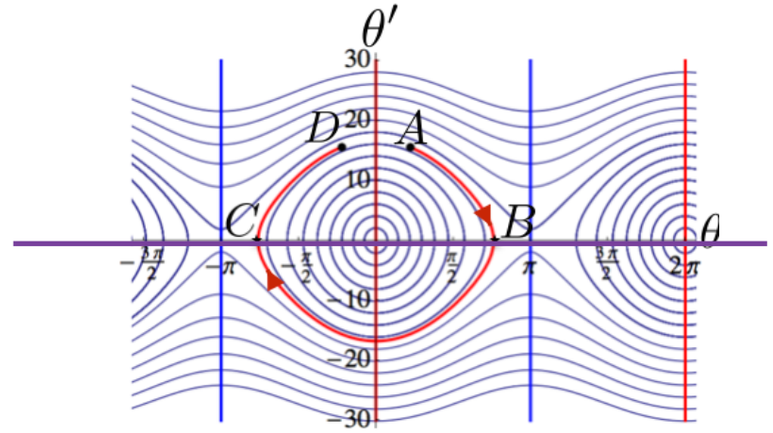




I=1
Stable

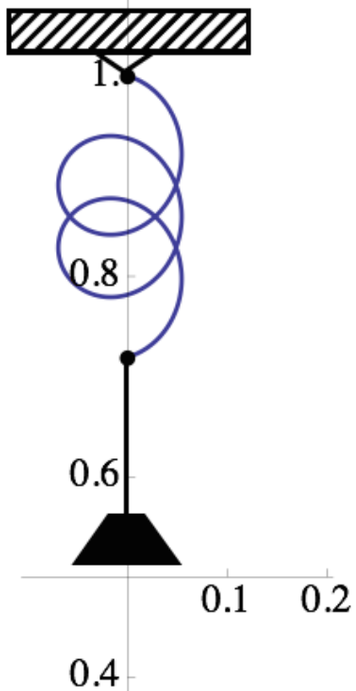
K=0
Dirichlet stable

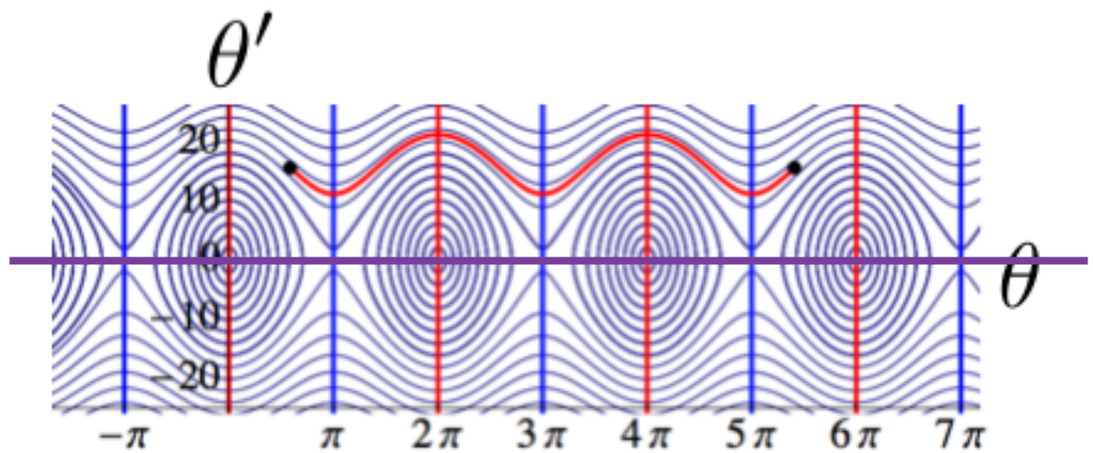




I=-1
Unstable

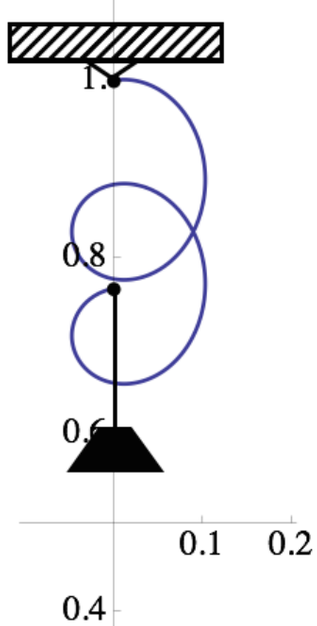
K=2
Dirichlet
unstable

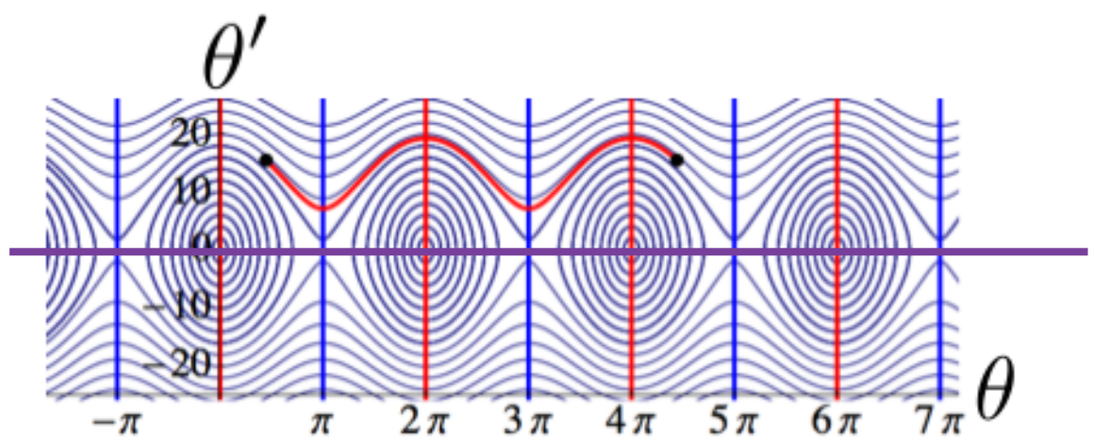




I=1
Stable

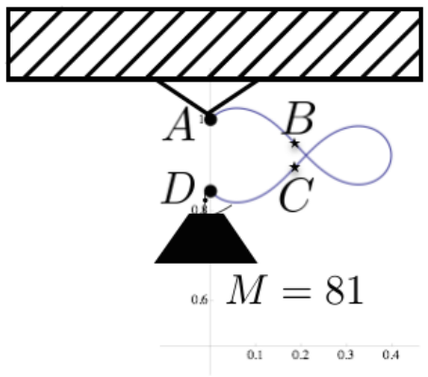
K=0
Dirichlet stable

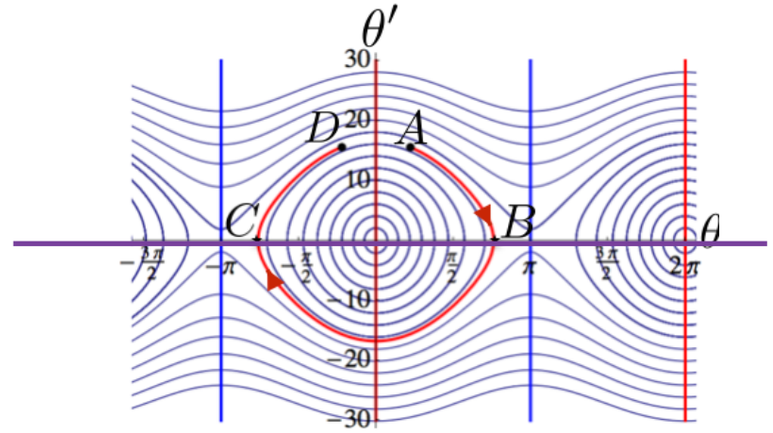




I=0
but **J**=0
Unstable

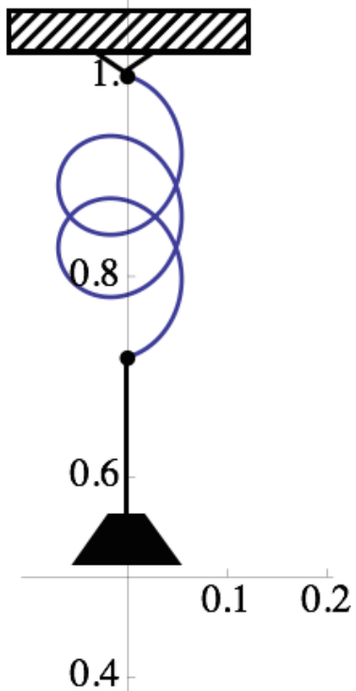
K=0
Dirichlet stable

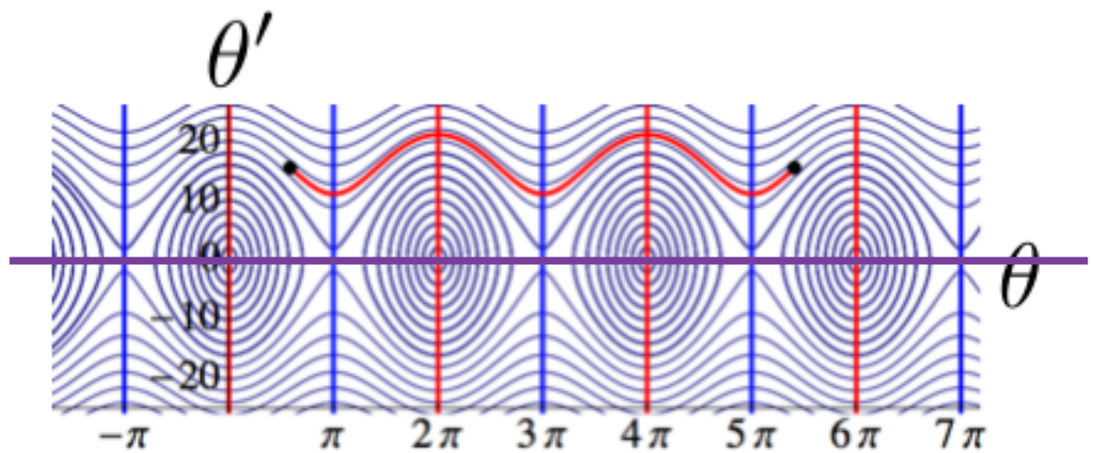




I=-1
Unstable

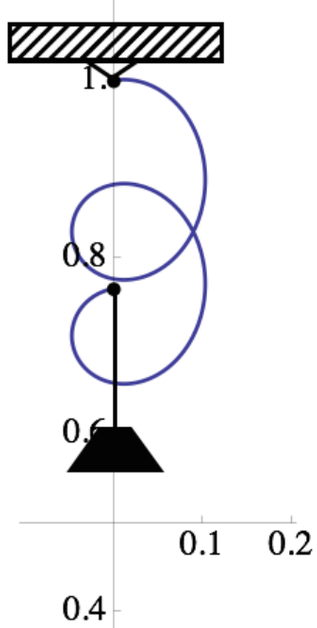
K=2
Dirichlet
unstable

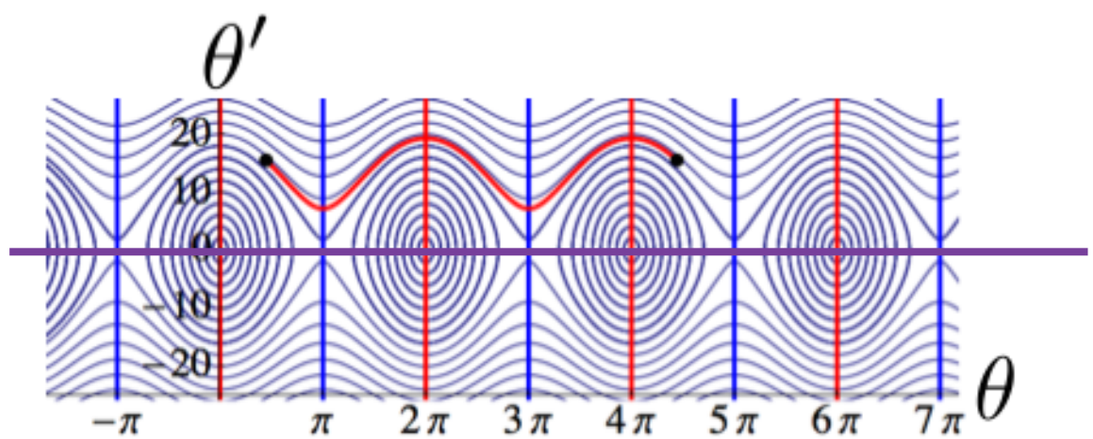




I=1
Stable

K=0
Dirichlet stable

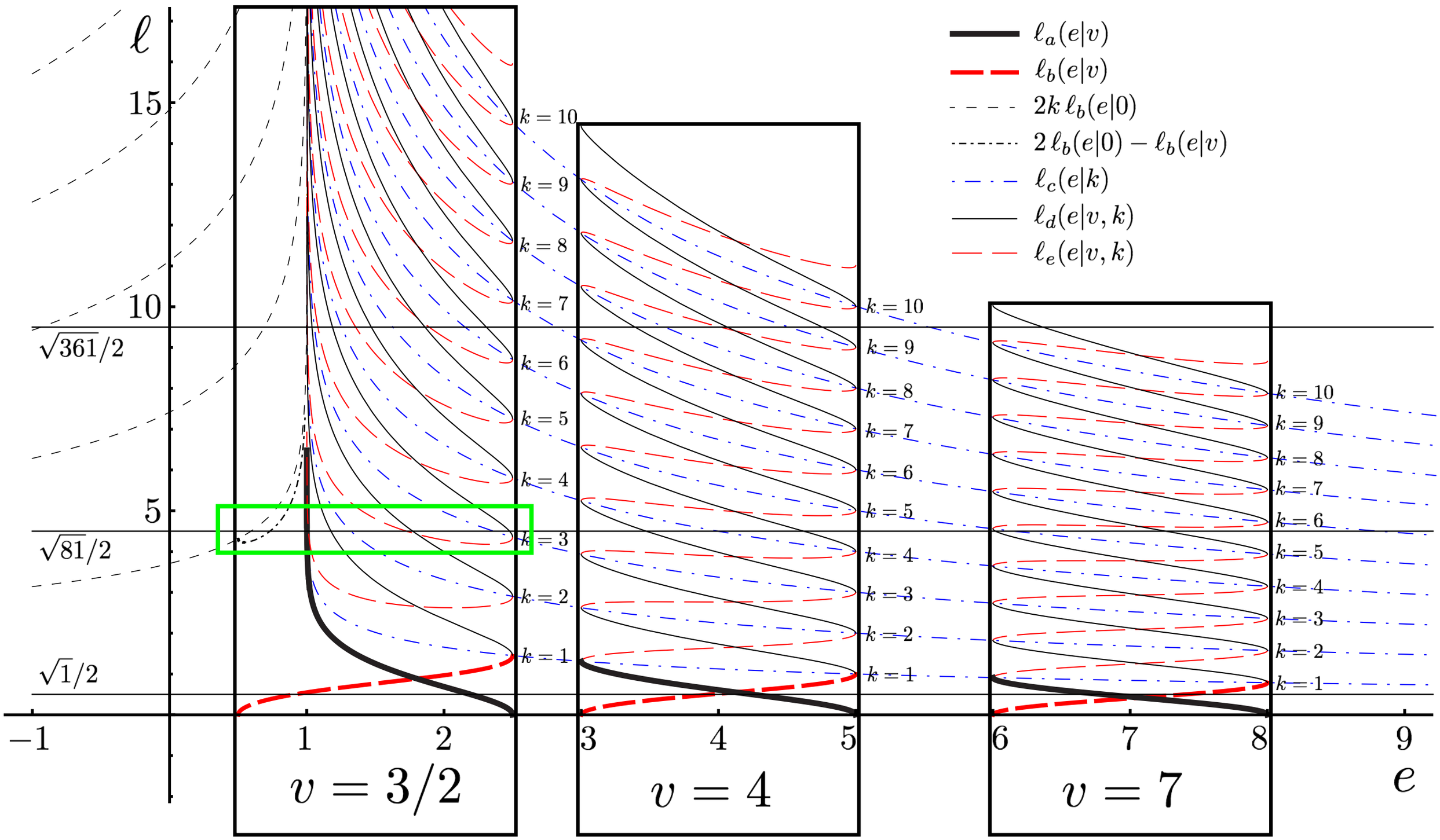




I=0
but **J**=0
Unstable

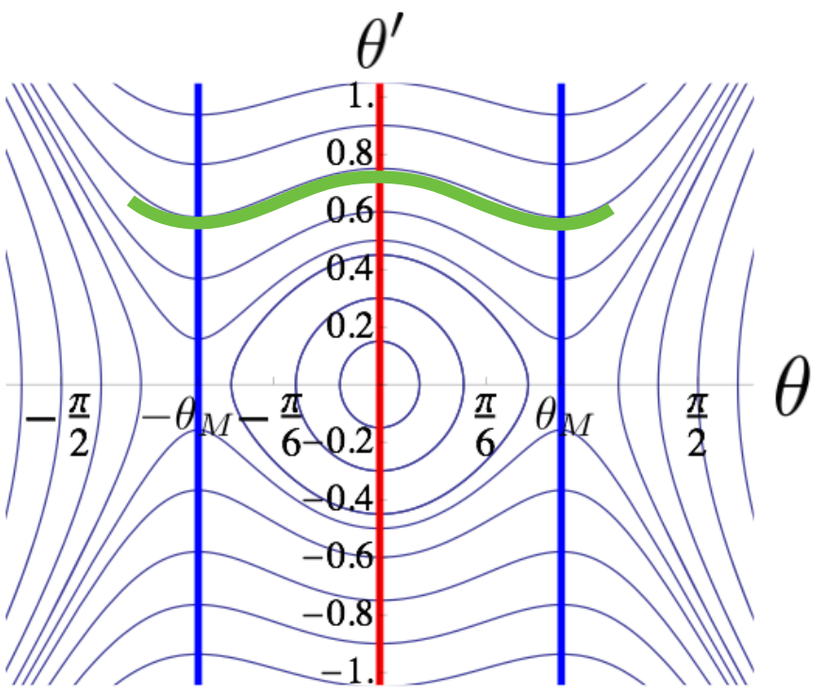
K=0
Dirichlet stable

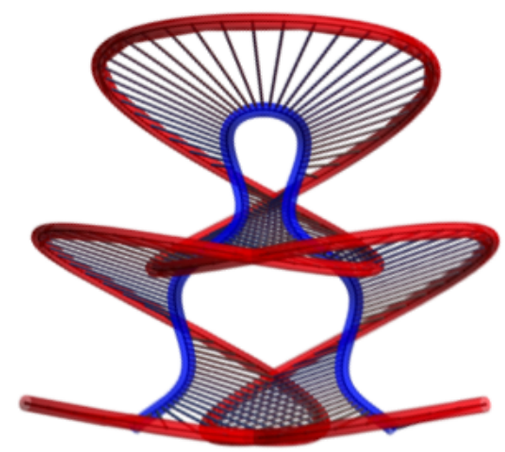
2 more stable cases and 2 more unstable ones (not shown)


ℓ
15
10
5
$\sqrt{361}/2$
$\sqrt{81}/2$
$\sqrt{1}/2$
$\ell_a(e|v)$
$\ell_b(e|v)$
$2k\,\ell_b(e|0)$
$2\,\ell_b(e|0) - \ell_b(e|v)$
$\ell_c(e|k)$
$\ell_d(e|v,k)$
$\ell_e(e|v,k)$
$k=10$
$k=9$
$k=8$
$k=7$
$k=6$
$k=5$
$k=4$
$k=3$
$k=2$
$k=1$
-1
1
2
3
4
5
6
7
8
9
e
$v = 3/2$
$v = 4$
$v = 7$

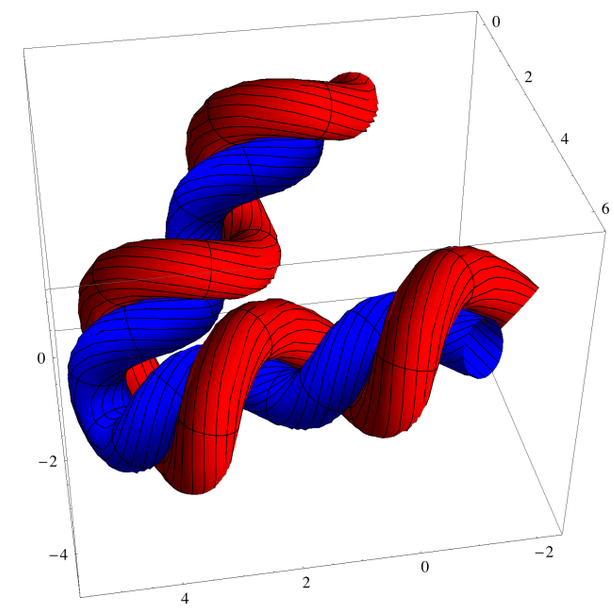
Conclusions

- I wish I were as creative as Keller





Bi-rods



Growing rods

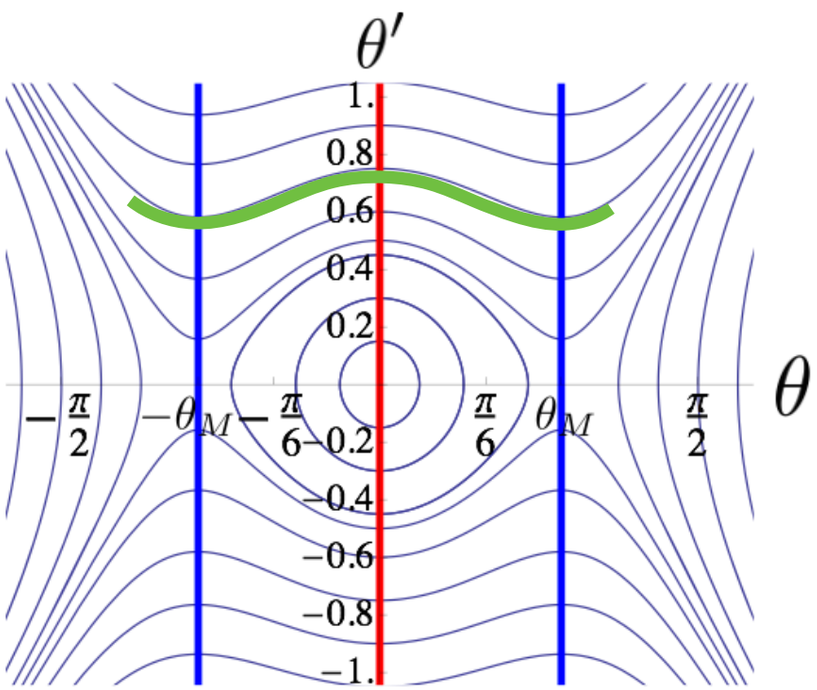
Thomas Lessinnes

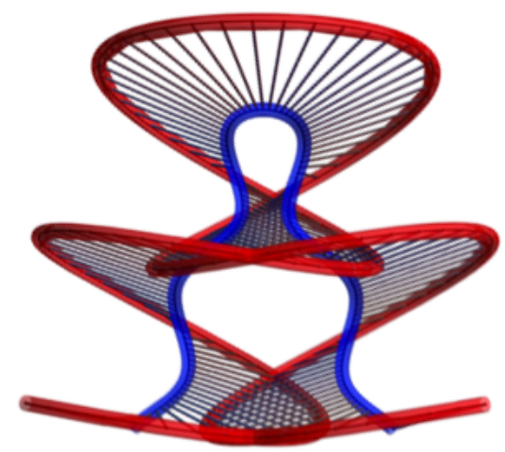
Geometric Stability (Nonlinearity- 2017) TL, AG

Stability of birods (SIAM J. Appl Math- 2016), TL, AG

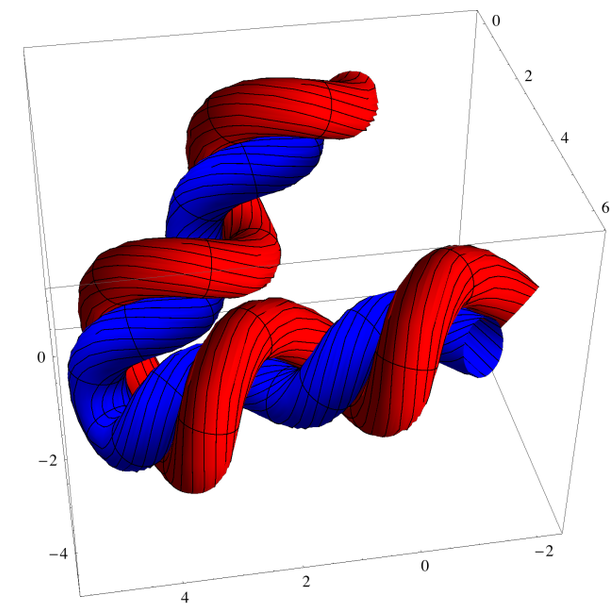
Conclusions

- I wish I were as creative as Keller
- If you can draw a solution, you know its stability





Bi-rods



Growing rods

Thomas Lessinnes

Geometric Stability (Nonlinearity- 2017) TL, AG

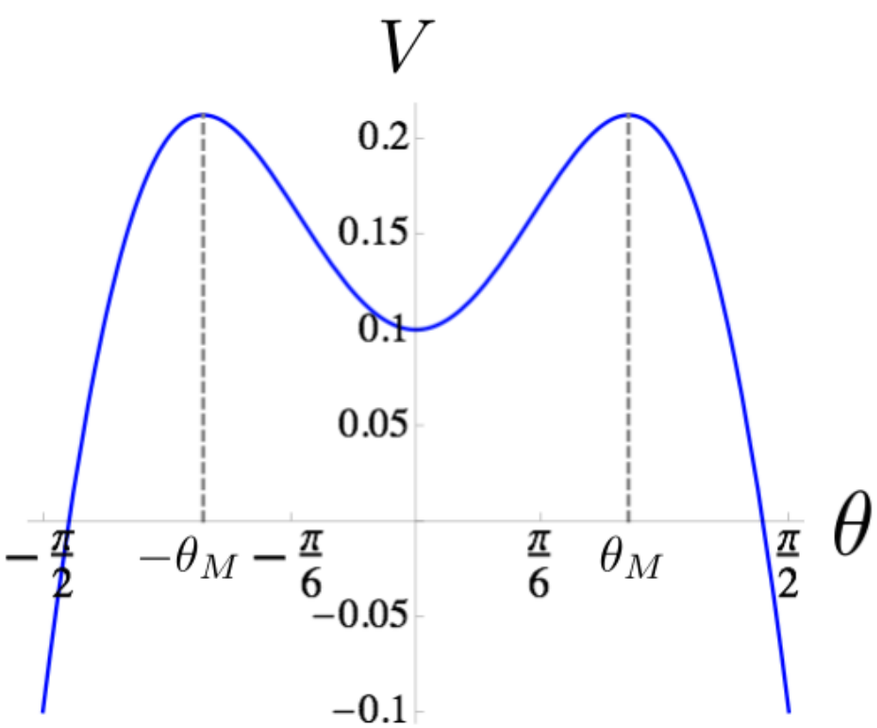
Stability of birods (SIAM J. Appl Math- 2016), TL, AG

The second variation

$$\delta^2 \mathscr{E}_\theta(\tau) = \int_a^b (\tau')^2 - \left.\frac{\mathrm{d}^2 V}{\mathrm{d}\theta^2}\right|_{\theta(s)} \tau^2 \;\; \mathrm{d}s > 0 \quad \forall \tau \in \mathcal{C}'([a,b])$$

$$\mathscr{L}(\theta, \theta') = \frac{(\theta' - A)^2}{2} - V(\theta)$$

$$\begin{cases} \theta'' + \dfrac{\mathrm{d}V}{\mathrm{d}\theta} = 0 \\ \theta'(a) = \theta'(b) = A. \end{cases}$$



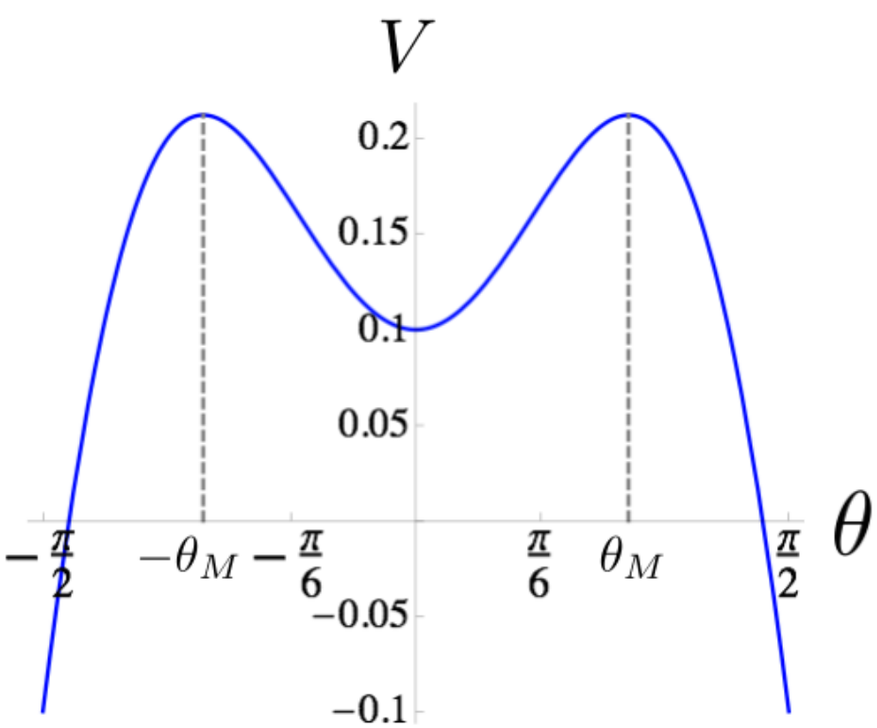
The second variation

$$\mathscr{L}(\theta,\theta') = \frac{(\theta'-A)^2}{2} - V(\theta)$$

$$\begin{cases} \theta'' + \dfrac{\mathrm{d}V}{\mathrm{d}\theta} = 0 \\ \theta'(a) = \theta'(b) = A. \end{cases}$$

$$\delta^2\mathscr{E}_\theta(\tau) = \int_a^b (\tau')^2 - \left.\frac{\mathrm{d}^2V}{\mathrm{d}\theta^2}\right|_{\theta(s)} \tau^2 \;\; \mathrm{d}s > 0 \quad \forall \tau \in \mathcal{C}'([a,b])$$

$$= \langle \tau | \mathcal{S}\tau \rangle > 0$$



The second variation

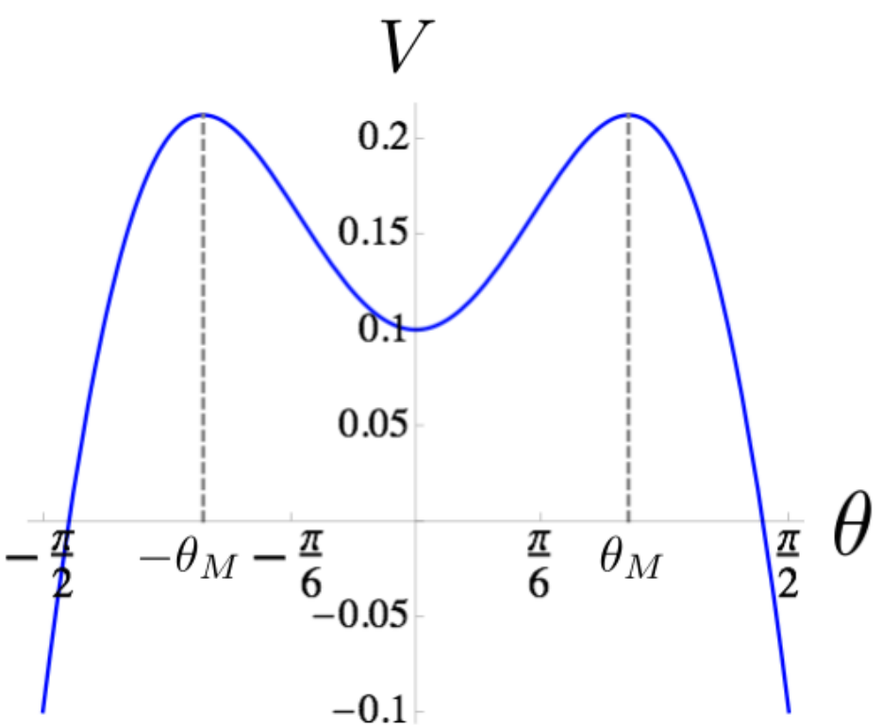
$$\mathscr{L}(\theta,\theta') = \frac{(\theta'-A)^2}{2} - V(\theta)$$

$$\begin{cases} \theta'' + \dfrac{\mathrm{d}V}{\mathrm{d}\theta} = 0 \\ \theta'(a) = \theta'(b) = A. \end{cases}$$

$$\delta^2\mathscr{E}_\theta(\tau) = \int_a^b (\tau')^2 - \left.\frac{\mathrm{d}^2V}{\mathrm{d}\theta^2}\right|_{\theta(s)} \tau^2 \quad \mathrm{d}s > 0 \quad \forall \tau \in \mathcal{C}'([a,b])$$

$$= \langle \tau | \mathcal{S}\tau \rangle > 0$$

$$\mathcal{S} = -\frac{\mathrm{d}^2}{\mathrm{d}s^2} + f(s) \qquad f(s) = -\left.\frac{\mathrm{d}^2V}{\mathrm{d}\theta^2}\right|_{\theta(s)}$$



The second variation

$$\mathscr{L}(\theta,\theta') = \frac{(\theta'-A)^2}{2} - V(\theta)$$

$$\begin{cases} \theta'' + \dfrac{\mathrm{d}V}{\mathrm{d}\theta} = 0 \\ \theta'(a) = \theta'(b) = A. \end{cases}$$

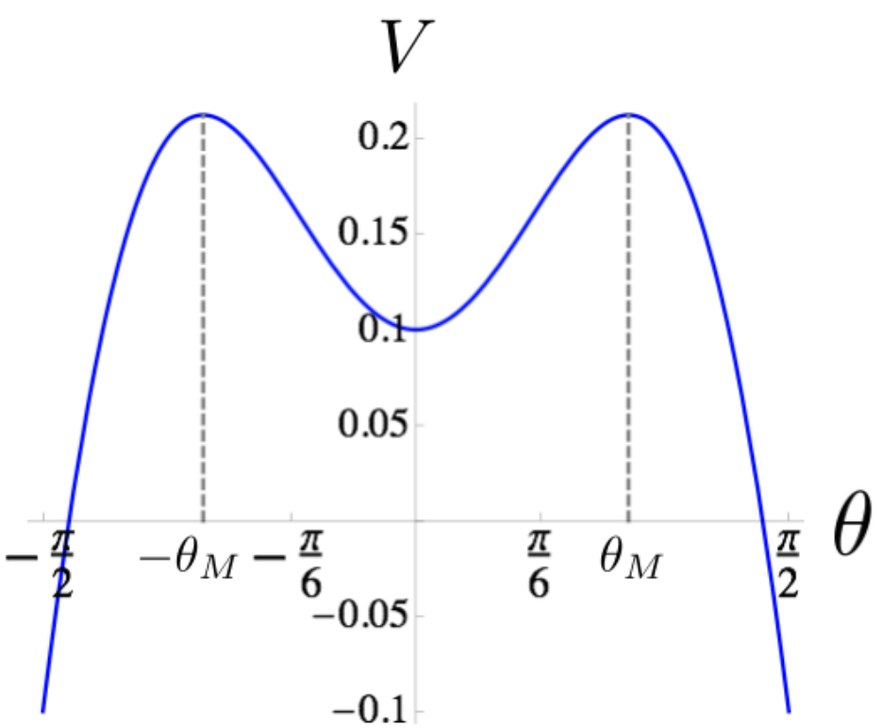
$$\delta^2\mathscr{E}_\theta(\tau) = \int_a^b (\tau')^2 - \left.\frac{\mathrm{d}^2V}{\mathrm{d}\theta^2}\right|_{\theta(s)} \tau^2 \;\; \mathrm{d}s > 0 \quad \forall \tau \in \mathcal{C}'([a,b])$$

$$= \langle \tau | \mathcal{S}\tau \rangle > 0$$

$$\mathcal{S} = -\frac{\mathrm{d}^2}{\mathrm{d}s^2} + f(s) \qquad f(s) = -\left.\frac{\mathrm{d}^2V}{\mathrm{d}\theta^2}\right|_{\theta(s)}$$

Sturm-Liouville problem.

$$\mathcal{S}\tau = \lambda\tau$$
$$\tau'(a) = \tau'(b) = 0.$$



Stability iff λ>0

The second variation

$$\mathscr{L}(\theta,\theta') = \frac{(\theta'-A)^2}{2} - V(\theta)$$

$$\begin{cases} \theta'' + \dfrac{\mathrm{d}V}{\mathrm{d}\theta} = 0 \\ \theta'(a) = \theta'(b) = A. \end{cases}$$

$$\delta^2\mathscr{E}_\theta(\tau) = \int_a^b (\tau')^2 - \left.\frac{\mathrm{d}^2V}{\mathrm{d}\theta^2}\right|_{\theta(s)} \tau^2 \;\; \mathrm{d}s > 0 \quad \forall \tau \in \mathcal{C}'([a,b])$$

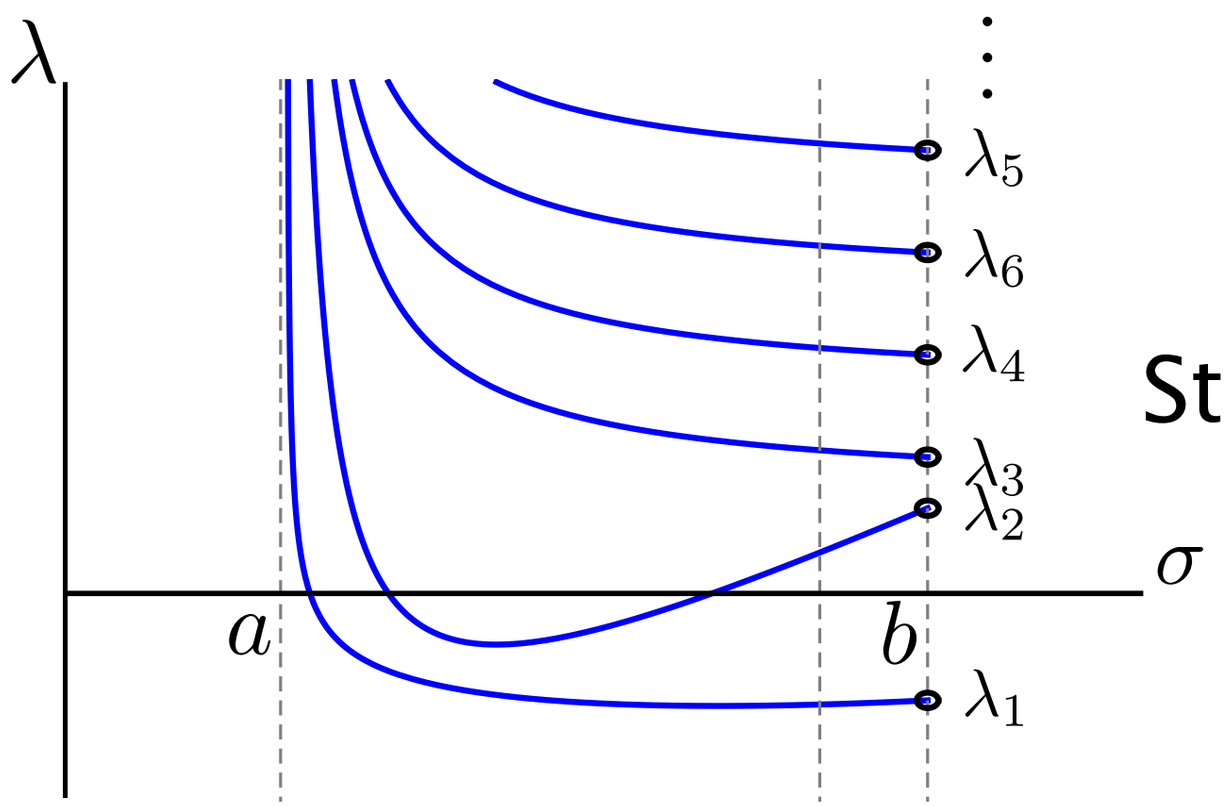
$$= \langle \tau | \mathcal{S}\tau \rangle > 0 \quad \forall \tau \in \mathcal{D}'([a,b])$$

$$\mathcal{S} = -\frac{\mathrm{d}^2}{\mathrm{d}s^2} + f(s) \qquad f(s) = -\left.\frac{\mathrm{d}^2V}{\mathrm{d}\theta^2}\right|_{\theta(s)}$$

Sturm-Liouville problem.

$$\mathcal{S}\tau = \lambda\tau$$

$$\tau'(a) = \tau'(b) = 0.$$



Stability iff λ>0

The second variation

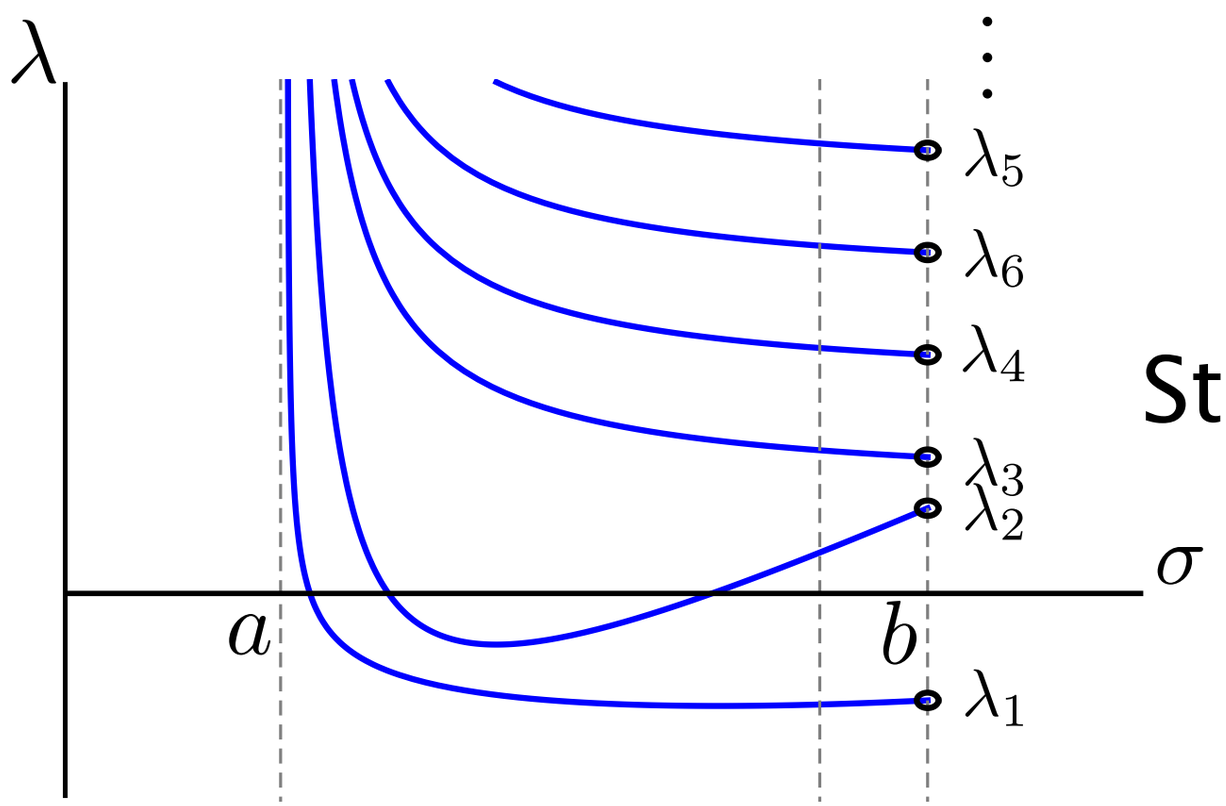
$$\mathscr{L}(\theta,\theta') = \frac{(\theta'-A)^2}{2} - V(\theta)$$

$$\begin{cases} \theta'' + \dfrac{\mathrm{d}V}{\mathrm{d}\theta} = 0 \\ \theta'(a) = \theta'(b) = A. \end{cases}$$

$$\delta^2\mathscr{E}_\theta(\tau) = \int_a^b (\tau')^2 - \left.\frac{\mathrm{d}^2V}{\mathrm{d}\theta^2}\right|_{\theta(s)} \tau^2 \;\; \mathrm{d}s > 0 \quad \forall \tau \in \mathcal{C}'([a,b])$$

$$= \langle \tau | \mathcal{S}\tau \rangle > 0 \quad \forall \tau \in \mathcal{D}'([a,b])$$

$$\mathcal{S} = -\frac{\mathrm{d}^2}{\mathrm{d}s^2} + f(s) \qquad f(s) = -\left.\frac{\mathrm{d}^2V}{\mathrm{d}\theta^2}\right|_{\theta(s)}$$



Sturm-Liouville problem.

$$\mathcal{S}\tau = \lambda\tau$$
$$\tau'(a) = \tau'(b) = 0.$$

Stability iff λ>0

First auxiliary problem

$$\mathcal{S}\tau = \lambda\tau$$
$$\tau'(a) = \tau'(\sigma) = 0.$$

The second variation

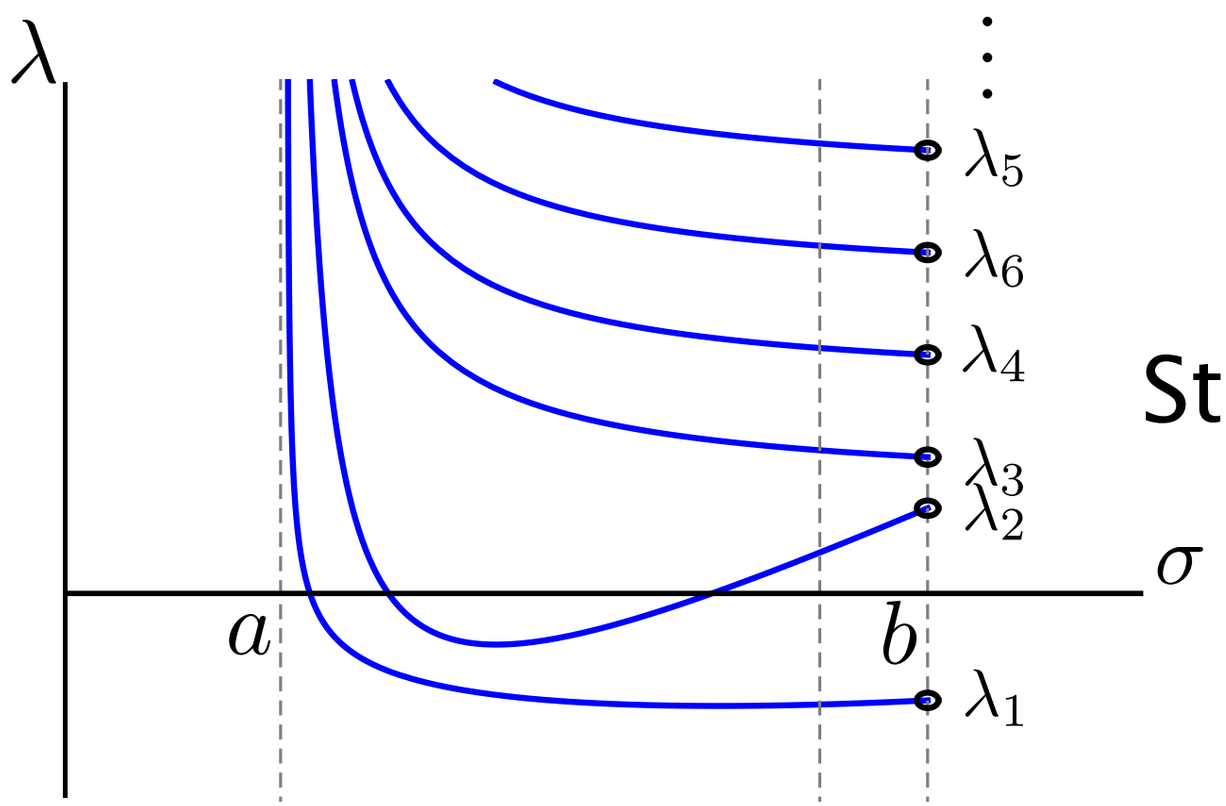
$$\mathscr{L}(\theta,\theta') = \frac{(\theta'-A)^2}{2} - V(\theta)$$

$$\begin{cases} \theta'' + \dfrac{\mathrm{d}V}{\mathrm{d}\theta} = 0 \\ \theta'(a) = \theta'(b) = A. \end{cases}$$

$$\delta^2\mathscr{E}_\theta(\tau) = \int_a^b (\tau')^2 - \left.\frac{\mathrm{d}^2V}{\mathrm{d}\theta^2}\right|_{\theta(s)} \tau^2 \;\; \mathrm{d}s > 0 \quad \forall \tau \in \mathcal{C}'([a,b])$$

$$= \langle \tau | \mathcal{S}\tau \rangle > 0 \quad \forall \tau \in \mathcal{D}'([a,b])$$

$$\mathcal{S} = -\frac{\mathrm{d}^2}{\mathrm{d}s^2} + f(s) \qquad f(s) = -\left.\frac{\mathrm{d}^2V}{\mathrm{d}\theta^2}\right|_{\theta(s)}$$



Sturm-Liouville problem.

$$\mathcal{S}\tau = \lambda\tau$$
$$\tau'(a) = \tau'(b) = 0.$$

Stability iff λ>0

First auxiliary problem

$$\mathcal{S}\tau = \lambda\tau$$
$$\tau'(a) = \boxed{\tau'(\sigma)} = 0.$$

The second variation

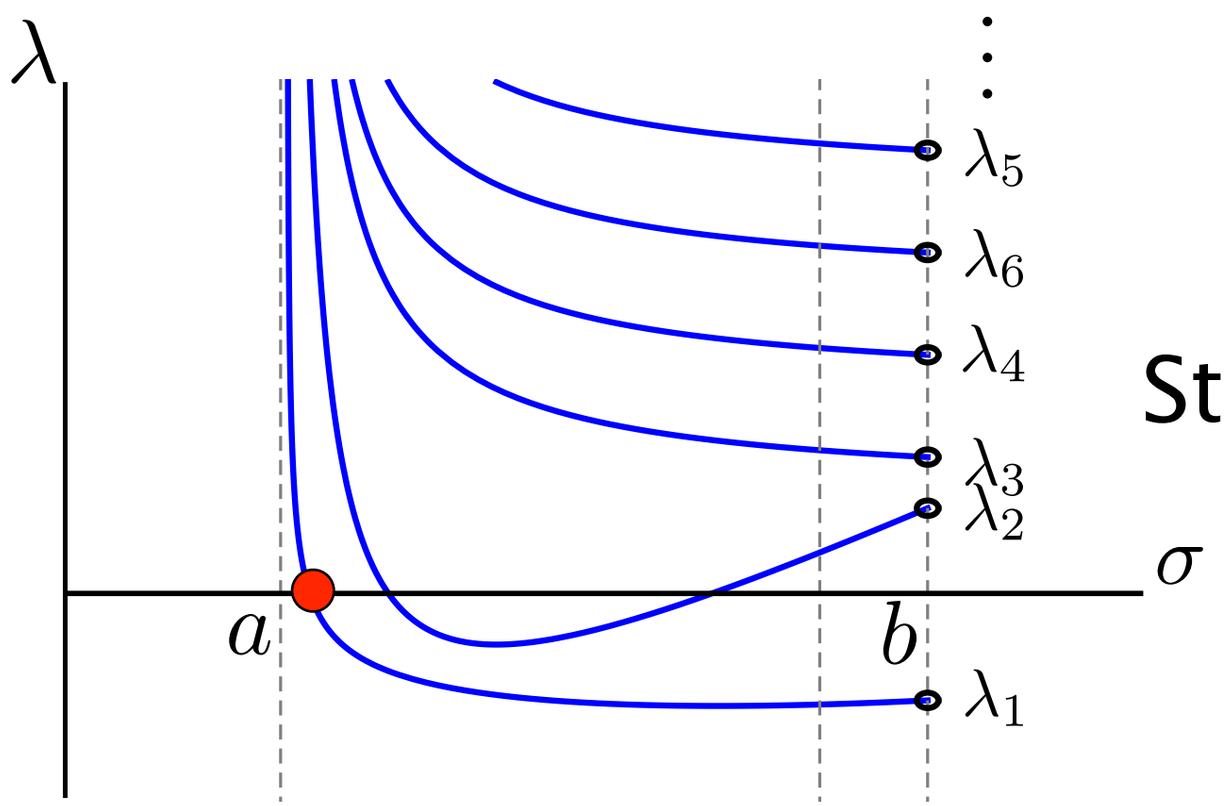
$$\mathscr{L}(\theta,\theta') = \frac{(\theta'-A)^2}{2} - V(\theta)$$

$$\begin{cases} \theta'' + \dfrac{\mathrm{d}V}{\mathrm{d}\theta} = 0 \\ \theta'(a) = \theta'(b) = A. \end{cases}$$

$$\delta^2\mathscr{E}_\theta(\tau) = \int_a^b (\tau')^2 - \left.\frac{\mathrm{d}^2V}{\mathrm{d}\theta^2}\right|_{\theta(s)} \tau^2 \;\; \mathrm{d}s > 0 \quad \forall \tau \in \mathcal{C}'([a,b])$$

$$= \langle \tau | \mathcal{S}\tau \rangle > 0 \quad \forall \tau \in \mathcal{D}'([a,b])$$

$$\mathcal{S} = -\frac{\mathrm{d}^2}{\mathrm{d}s^2} + f(s) \qquad f(s) = -\left.\frac{\mathrm{d}^2V}{\mathrm{d}\theta^2}\right|_{\theta(s)}$$



Sturm-Liouville problem.

$$\mathcal{S}\tau = \lambda\tau$$
$$\tau'(a) = \tau'(b) = 0.$$

Stability iff λ>0

First auxiliary problem

$$\mathcal{S}\tau = \lambda\tau$$
$$\tau'(a) = \boxed{\tau'(\sigma)} = 0.$$

The second variation

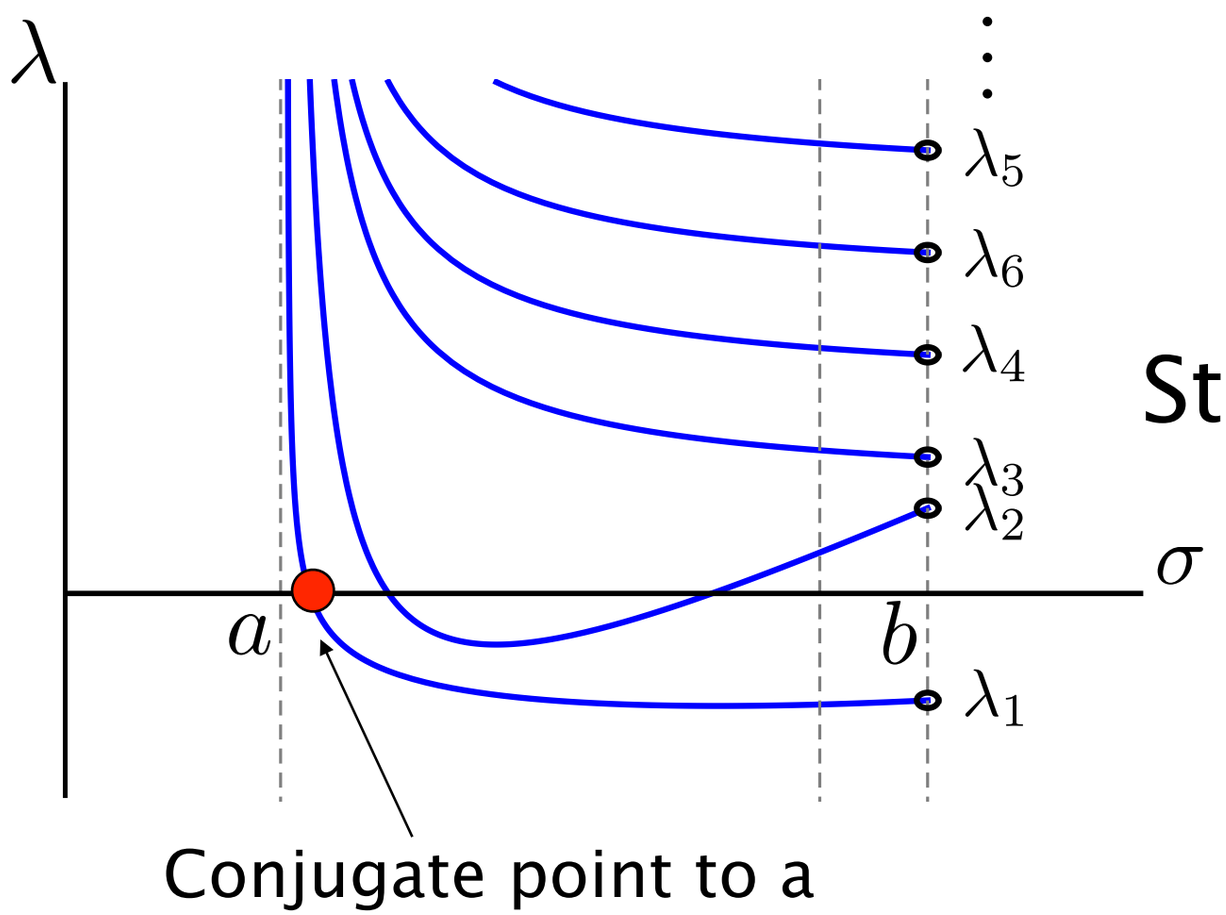
$$\mathscr{L}(\theta,\theta') = \frac{(\theta'-A)^2}{2} - V(\theta)$$

$$\begin{cases} \theta'' + \dfrac{\mathrm{d}V}{\mathrm{d}\theta} = 0 \\ \theta'(a) = \theta'(b) = A. \end{cases}$$

$$\delta^2\mathscr{E}_\theta(\tau) = \int_a^b (\tau')^2 - \left.\frac{\mathrm{d}^2V}{\mathrm{d}\theta^2}\right|_{\theta(s)} \tau^2 \;\; \mathrm{d}s > 0 \quad \forall \tau \in \mathcal{C}'([a,b])$$

$$= \langle \tau | \mathcal{S}\tau \rangle > 0 \quad \forall \tau \in \mathcal{D}'([a,b])$$

$$\mathcal{S} = -\frac{\mathrm{d}^2}{\mathrm{d}s^2} + f(s) \qquad f(s) = -\left.\frac{\mathrm{d}^2V}{\mathrm{d}\theta^2}\right|_{\theta(s)}$$



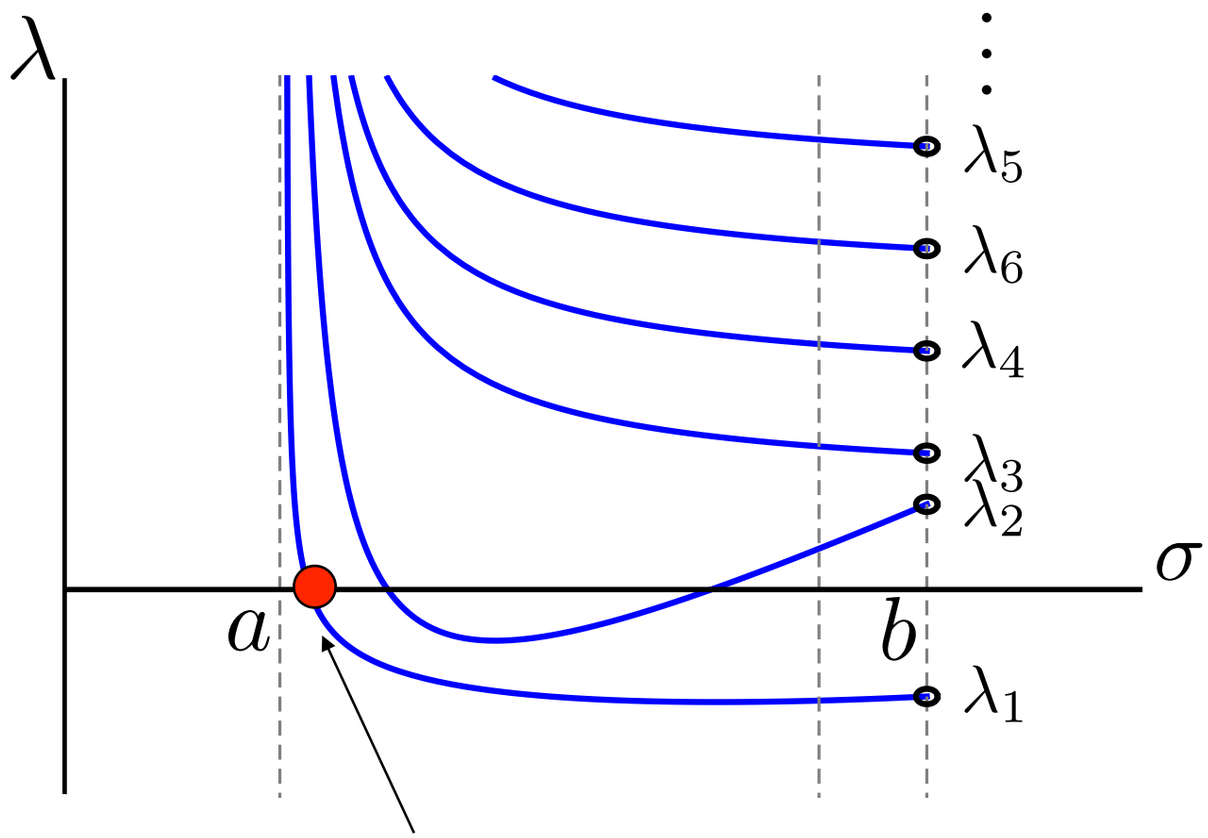
Sturm-Liouville problem.

$$\mathcal{S}\tau = \lambda\tau$$
$$\tau'(a) = \tau'(b) = 0.$$

Stability iff λ>0

First auxiliary problem

$$\mathcal{S}\tau = \lambda\tau$$
$$\tau'(a) = \boxed{\tau'(\sigma)} = 0.$$



Conjugate point to a

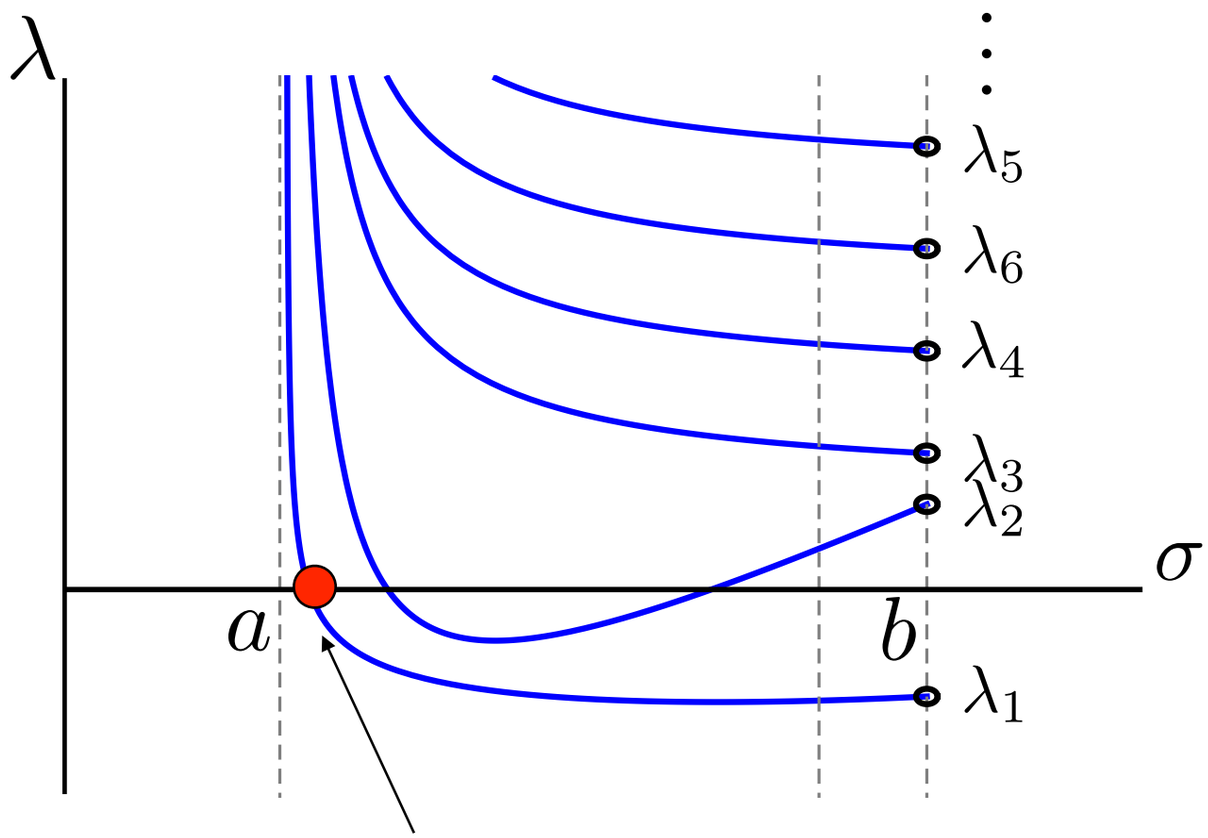
Sturm-Liouville problem.

$$\mathcal{S}\tau = \lambda\tau$$
$$\tau'(a) = \tau'(b) = 0.$$

Stability iff λ>0

First auxiliary problem

$$\mathcal{S}\tau = \lambda\tau$$
$$\tau'(a) = \boxed{\tau'(\sigma)} = 0.$$



Conjugate point to a

Sturm-Liouville problem.

$$\mathcal{S}\tau = \lambda\tau$$
$$\tau'(a) = \tau'(b) = 0.$$

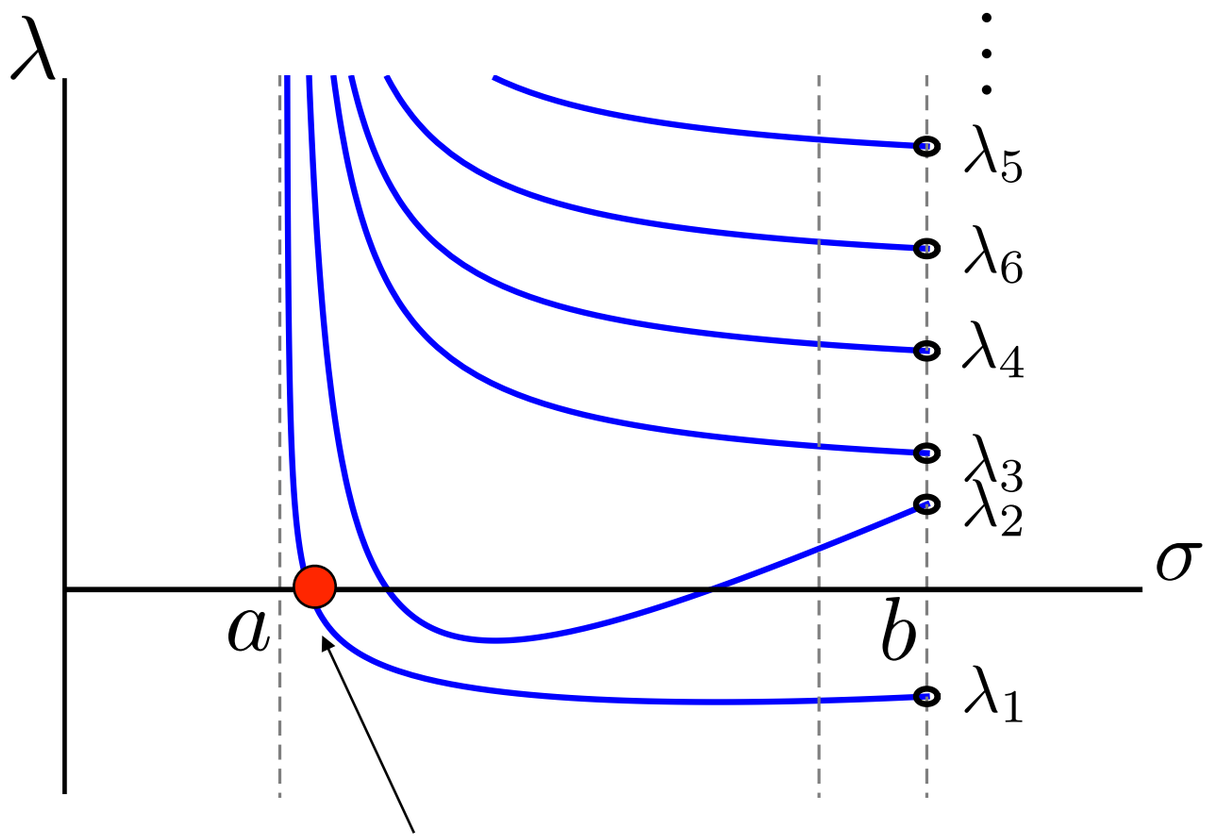
Stability iff λ>0

First auxiliary problem

$$\mathcal{S}\tau = \lambda\tau$$
$$\tau'(a) = \boxed{\tau'(\sigma)} = 0.$$

Second auxiliary problem

$$\mathcal{S}h = 0,$$
$$h(a) = 1,\ h'(a) = 0$$



Conjugate point to a

Sturm-Liouville problem.

$$\mathcal{S}\tau = \lambda\tau$$
$$\tau'(a) = \tau'(b) = 0.$$

Stability iff λ>0

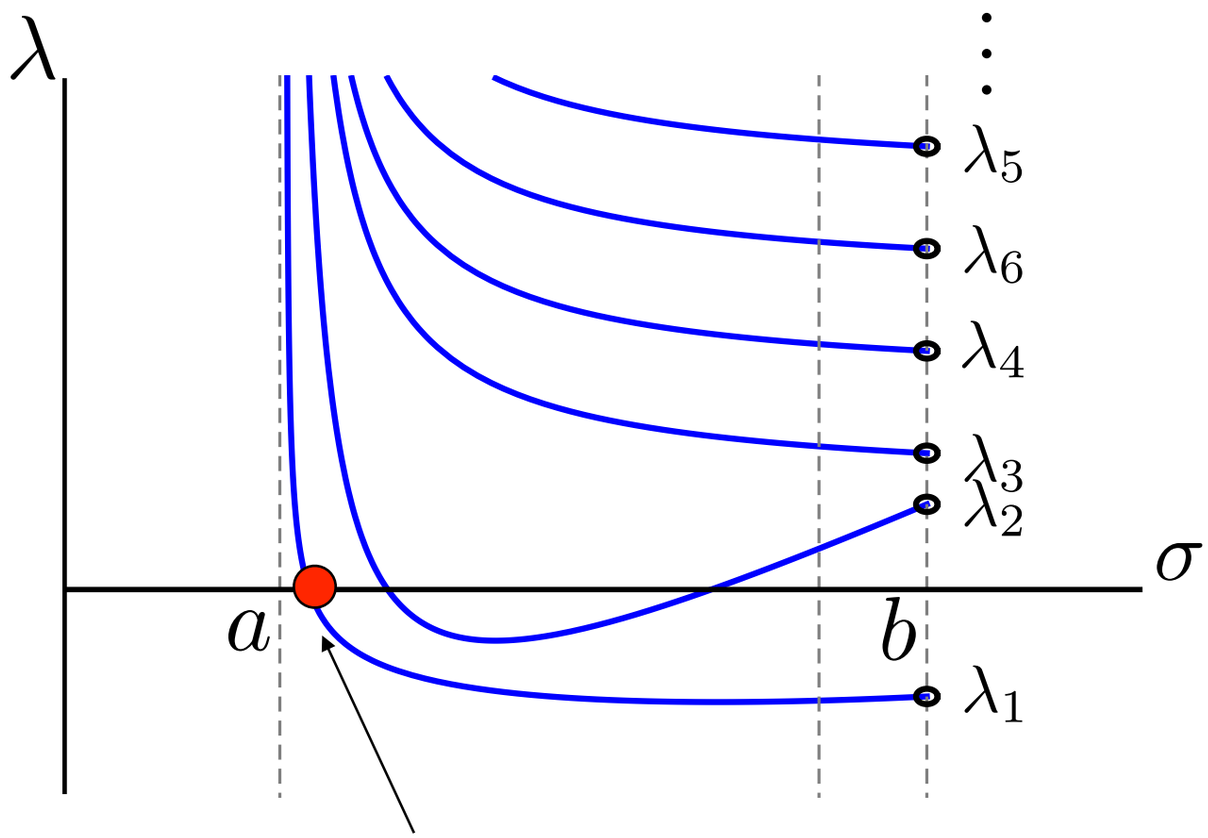
First auxiliary problem

$$\mathcal{S}\tau = \lambda\tau$$
$$\tau'(a) = \boxed{\tau'(\sigma)} = 0.$$

Second auxiliary problem

$$\mathcal{S}h = 0,$$
$$h(a) = 1,\ h'(a) = 0$$

$\Rightarrow$ The roots of h' are the conjugate points



Conjugate point to a

Sturm-Liouville problem.

$$\mathcal{S}\tau = \lambda\tau$$
$$\tau'(a) = \tau'(b) = 0.$$

Stability iff λ>0

First auxiliary problem

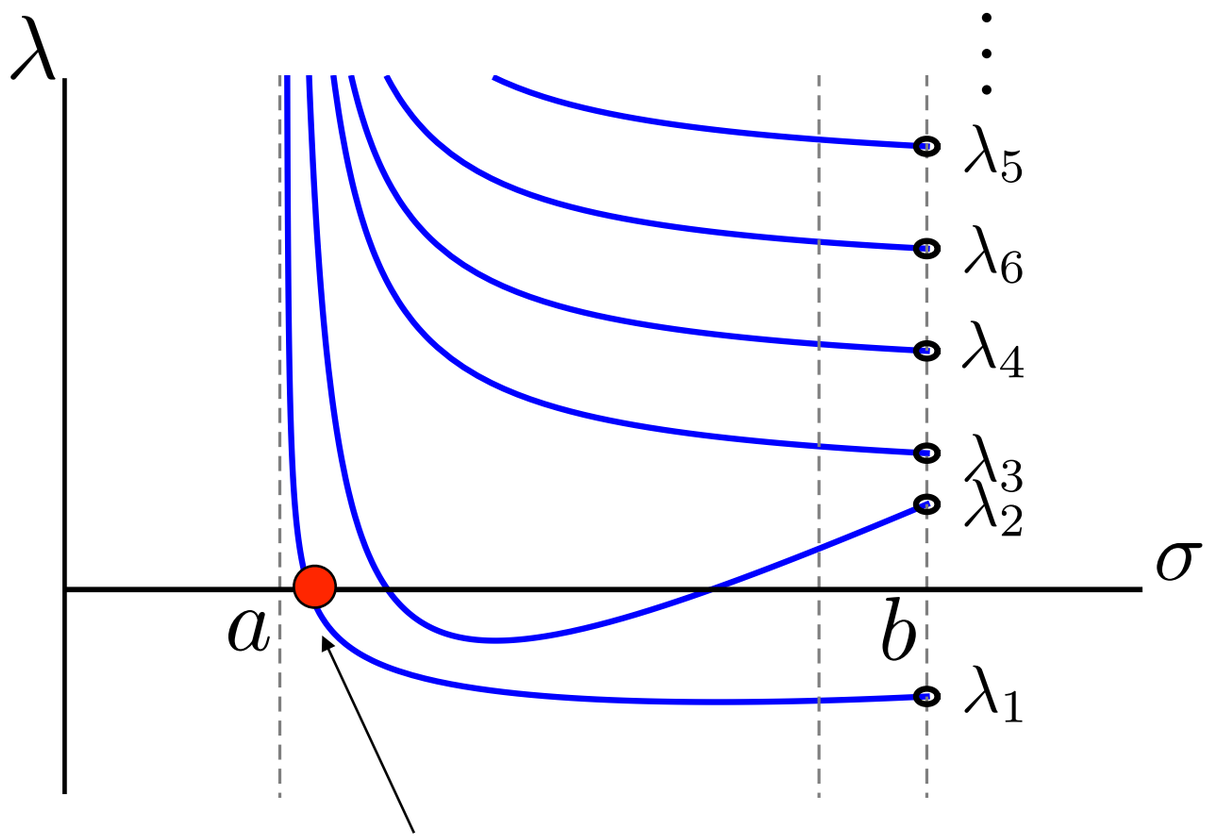
$$\mathcal{S}\tau = \lambda\tau$$
$$\tau'(a) = \boxed{\tau'(\sigma)} = 0.$$

Second auxiliary problem

$$\mathcal{S}h = 0,$$
$$h(a) = 1,\ h'(a) = 0$$

$\Rightarrow$ The roots of h' are the conjugate points

$\Leftrightarrow$ The roots of θ'' are the conjugate points



Conjugate point to a

Sturm-Liouville problem.

$$\mathcal{S}\tau = \lambda\tau$$
$$\tau'(a) = \tau'(b) = 0.$$

Stability iff λ>0

First auxiliary problem

$$\mathcal{S}\tau = \lambda\tau$$
$$\tau'(a) = \boxed{\tau'(\sigma)} = 0.$$

Second auxiliary problem

$$\mathcal{S}h = 0,$$
$$h(a) = 1,\ h'(a) = 0$$

$\Rightarrow$ The roots of h' are the conjugate points

$\Leftrightarrow$ The roots of θ'' are the conjugate points

$\Leftrightarrow$ The extrema of V give the conjugate points