Mathematics of the Faraday Cage

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On a metal shell the potential is constant.

Thus the potential inside is constant, so the field inside is zero.

Faraday observed in 1836 that the same holds (nearly) for a metal mesh.

This effect is used in shielding (e.g. microwave oven).
I got interested in this problem because of Trefethen. He got interested because of an analogy.

The trapezoidal quadrature rule is exponentially accurate for periodic analytic integrands. Perhaps the mathematics of the Faraday cage is analogous, and the shielding effect is exponential?

In this analogy, trapezoidal quadrature points corresponds to the cross-sections of wires of a Faraday cage.
Feynman’s *Lecture Notes* gives one of the few treatments, appearing to confirm the intuition that the effect is exponential.

"The method we have just developed can be used to explain why electrostatic shielding by means of a screen is often just as good as with a solid metal sheet. Except within a distance from the screen a few times the spacing of the screen wires, the fields inside a closed screen are zero.

The intuition is wrong. The shielding is much weaker, just linear in the mesh spacing."
Outline

1. Electrostatic Faraday cage: numerics

2. Homogenised model via multiple scales

3. Point charges model via constrained quadratic optimisation
1. Electrostatic Faraday cage: numerics

\[ \nabla^2 \phi = 0 \]

\[ \phi = \phi_0 \text{ on } n \text{ disks of radius } r \]

Boundary conditions

\[ \phi = \phi_0 = \text{unknown constant on disks radius } r \text{ at the } n\text{th roots of unity} \]

\[ \phi = \log |z - z_s| + O(1) \text{ as } z \to z_s \]

\[ \phi = \log |z| + o(1) \text{ as } z \to z_s \text{ (zero total charge on the disks)} \]
Dependence on $r$: logarithmic ($n = 12$)
Dependence on $n$: linear ($r = 0.01$)
2. Multiple scales analysis

Inner region $(x, y) = (r\xi, r\eta)$.

$$\phi_{\xi\xi} + \phi_{\eta\eta} = 0 \quad \text{for} \quad \xi^2 + \eta^2 > 1,$$

$$\phi = \phi_0 \quad \text{on} \quad \xi^2 + \eta^2 = 1.$$

Complex potential

$$w = \phi + i\psi = \phi_0 + A \log \zeta, \quad \zeta = \xi + i\eta.$$
2. Multiple scales analysis

Middle region \((x, y) = (\epsilon X, \epsilon X)\).

\[
\phi_{XX} + \phi_{YY} = 2\pi C \delta(X, Y) \quad \text{for } |Y| < 1/2,
\]

\[
\phi_Y = 0 \quad \text{on } Y = \pm 1/2.
\]

\(C = \) charge on the wire.

\[
w = B + C \log (\sinh (\pi Z)).
\]

Inner region \((x, y) = (r\xi, r\eta)\).

\[
w = \phi_0 + A \log \zeta.
\]
Matching

Middle region \((x, y) = (\epsilon X, \epsilon X)\).

\[ w \sim B + C \log(\pi Z) \quad \text{as} \quad Z \to 0. \]

Inner region \((x, y) = (r\xi, r\eta)\).

\[ w \sim \phi_0 + A \log \frac{1}{\delta} + A \log Z. \]
Matching gives

\[ \phi_0 + A \log \frac{1}{\delta} = B + C \log \pi, \]

\[ A = C. \]

**Middle region** \((x, y) = (\epsilon X, \epsilon X)\).

\[ w \sim B + C \log(\pi Z) \text{ as } Z \to 0. \]

**Inner region** \((x, y) = (r \xi, r \eta)\).

\[ w \sim \phi_0 + A \log \frac{1}{\delta} + A \log Z. \]
Matching

Middle region \((x, y) = (\epsilon X, \epsilon X)\).

\[ \phi \sim \pm C \pi X - C \log 2 + B \text{ as } X \to \pm \infty. \]
Matching

**Outer region**

Matching gives

$$\left[ \frac{\partial \phi}{\partial n} \right]_{\Gamma^+} = \frac{2\pi C}{\epsilon}, \quad \phi(\Gamma) = B - C \log 2.$$ 

**Middle region** $(x, y) = (\epsilon X, \epsilon X)$.

$$\phi \sim \pm C \pi X - C \log 2 + B \text{ as } X \to \pm \infty.$$
Eliminating $A$, $B$ and $C$ gives

$$\left[ \frac{\partial \phi}{\partial n} \right]^+ = \alpha (\phi - \phi_0) \text{ on } \Gamma,$$

$$\alpha = \frac{2\pi}{\epsilon \log(\epsilon/(2\pi r))} = \frac{n}{\log(1/(rn))}.$$

$\alpha(\phi - \phi_0) = \text{charge density.}$  
$\alpha = \text{capacitance per unit length.}$

Solution of homogenised problem has

$$|\nabla \phi(0)| = \frac{2}{(2 + \alpha)|z_s|}.$$

Effective screening requires $\alpha \gg 1$, i.e.

$$r \gg \frac{e^{-n}}{n}.$$

Thus when there are many wires they can be very thin.
Dependence on $r (n = 12)$

- Actual
- Homogenized BC
Dependence on $n$ ($r = 0.01$)

![Diagram showing dependence on $n$ with $r = 0.01$ for $n = 10$, $n = 20$, and $n = 40$. The top row shows actual contours, and the bottom row shows homogenized BC contours.]
A disk at $z_k$ of radius $r \ll 1$ behaves like a point charge of some strength $q_k$. The challenge for Faraday cage analysis is, what is $q_k$? We can find these charges by energy minimisation.

Key observation: a charged disk has a self-energy of $-\frac{1}{2} q_k^2 \log r$. It takes work to force charge onto a small disk.
Find \( n \) charges \( q_k \), summing to 0, that minimise the total energy:

\[
E(q) = -\frac{1}{2} \sum_{k=1}^{n} q_k^2 \log r - \sum_{k=1}^{n} \sum_{j>k} q_j q_k \log |z_k - z_j| - \sum_{k=1}^{n} q_k \log |z_k - z_s|
\]

This is a constrained quadratic programming problem

\[
\text{minimise } E(q) = \frac{1}{2} q^T A q - f^T q, \quad c^T q = 0.
\]
Dependence on $n$ $(r = 0.01)$
Small charged disks can be approximated by point charges to high accuracy.
A charge distribution on a smooth curve can be approximated by point samples to high accuracy—the trapezoidal rule.

*So why doesn’t the Faraday cage provide strong screening?*

Because it provides an exponentially good approximation not to the zero field, but to the field associated with the homogenized boundary condition.
Finite wires, \( r = 0.01 \)

Point charges from sampling solution from homogenized BC

Point charges from sampling solution from perfectly conducting shell