Mathematics of the Faraday Cage

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Faraday Cage

On a metal shell the potential is constant.

Thus the potential inside is constant, so the field inside is zero.

Faraday observed in 1836 that the same holds (nearly) for a metal mesh.

This effect is used in shielding (e.g. microwave oven).

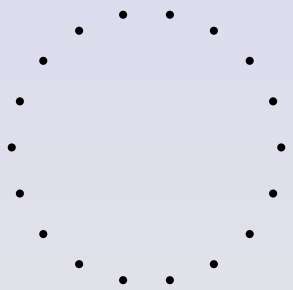
Photo from Museum of Science, Boston

Analogy

I got interested in this problem because of Trefethen.
He got interested because of an analogy.

The trapezoidal quadrature rule is exponentially accurate for periodic analytic integrands. Perhaps the mathematics of the Faraday cage is analogous, and the shielding effect is exponential?

In this analogy, trapezoidal quadrature points corresponds to the cross-sections of wires of a Faraday cage.



Electrostatic case

Feynman's *Lecture Notes* gives one of the few treatments, appearing to confirm the intuition that the effect is exponential.

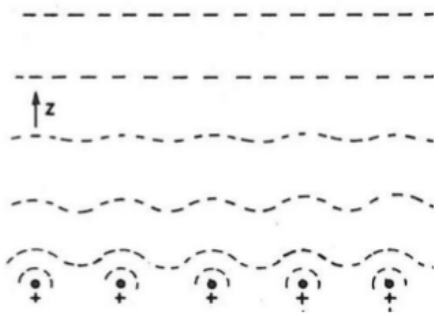
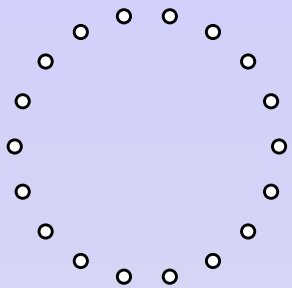


Fig. 7–8. Equipotential surfaces above a uniform grid of charged wires.

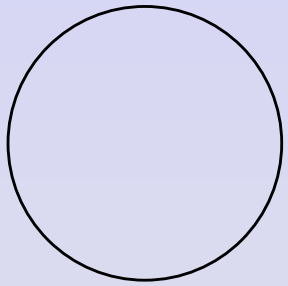
"The method we have just developed can be used to explain why electrostatic shielding by means of a screen is often just as good as with a solid metal sheet. Except within a distance from the screen a few times the spacing of the screen wires, the fields inside a closed screen are zero."

The intuition is wrong.
The shielding is much weaker, just linear in the mesh spacing.

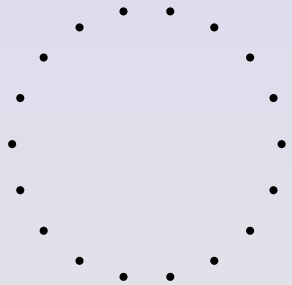
Outline



1. Electrostatic Faraday cage: numerics

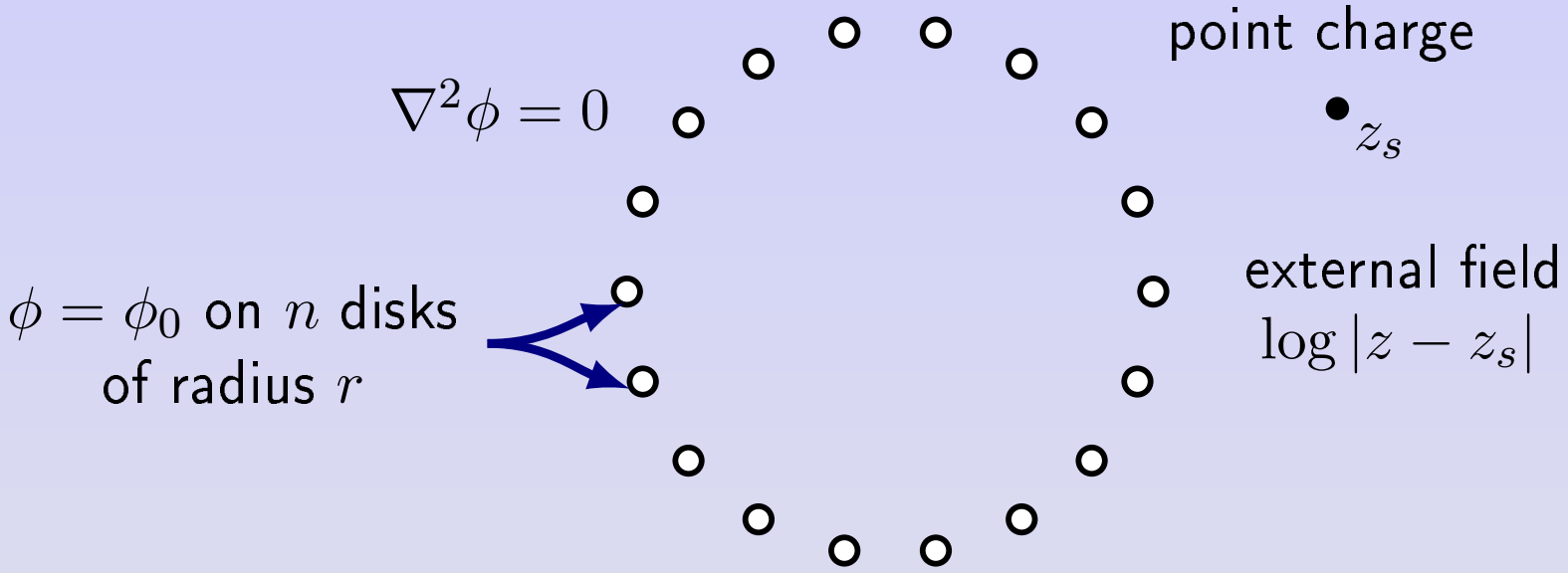


2. Homogenised model via multiple scales



3. Point charges model via constrained quadratic optimisation

1. Electrostatic Faraday cage: numerics



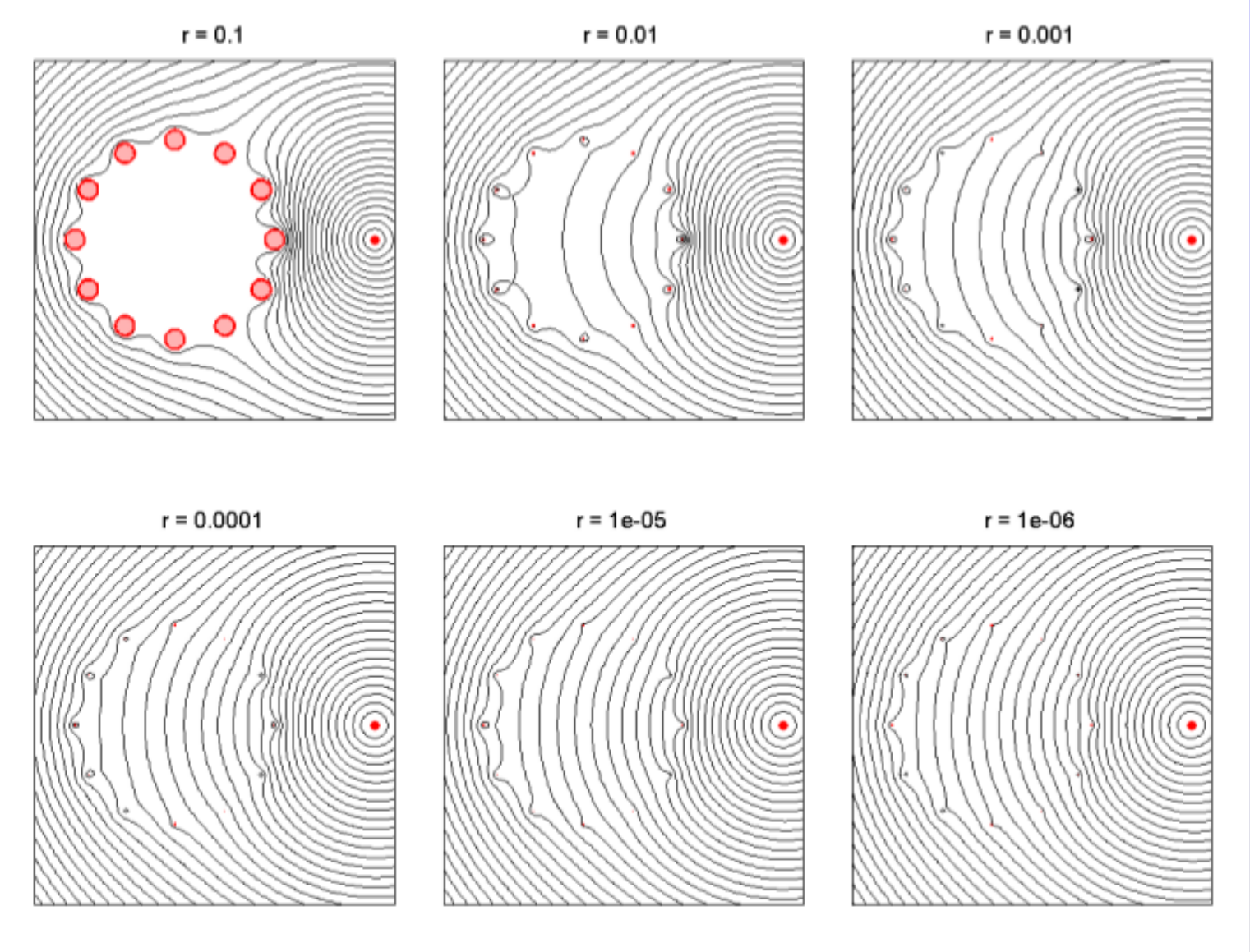
Boundary conditions

$\phi = \phi_0 =$ unknown constant on disks radius r at the nth roots of unity

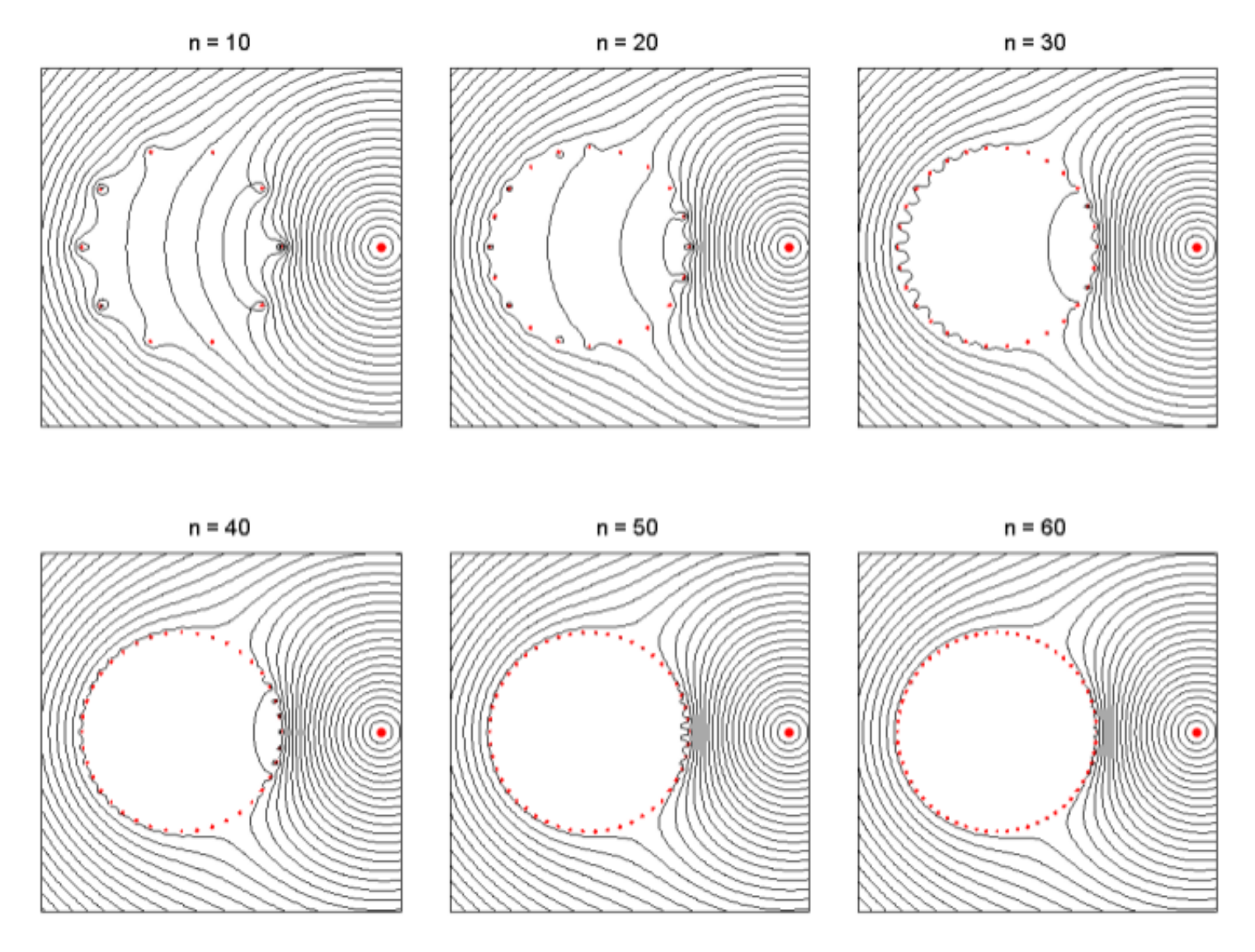
$\phi = \log|z - z_s| + O(1)$ as $z \to z_s$

$\phi = \log|z| + o(1)$ as $z \to z_s$ (zero total charge on the disks)

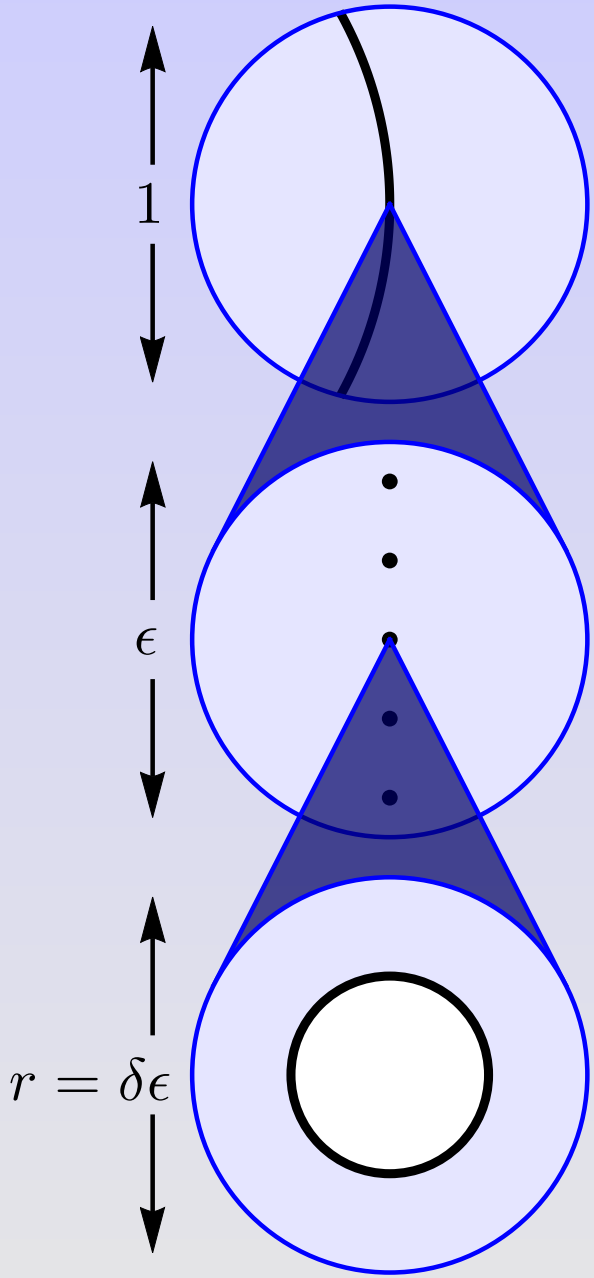
Dependence on r: logarithmic $(n = 12)$



Dependence on n: linear $(r = 0.01)$



2. Multiple scales analysis



Inner region $(x, y) = (r\xi, r\eta)$.

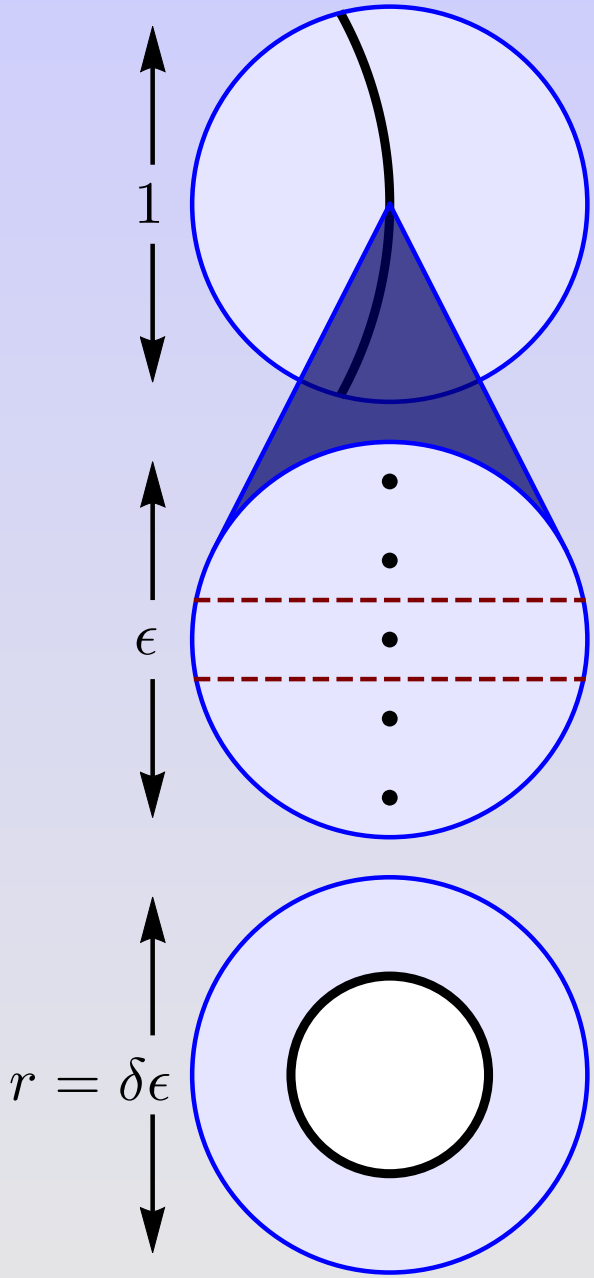
$$\phi_{\xi\xi} + \phi_{\eta\eta} = 0 \quad \text{for } \xi^2 + \eta^2 > 1,$$

$$\phi = \phi_0 \quad \text{on } \xi^2 + \eta^2 = 1.$$

Complex potential

$$w = \phi + \mathrm{i}\psi = \phi_0 + A \log \zeta, \quad \zeta = \xi + \mathrm{i}\eta.$$

2. Multiple scales analysis



Middle region $(x, y) = (\epsilon X, \epsilon X)$.

$$\phi_{XX} + \phi_{YY} = 2\pi C \delta(X, Y) \quad \text{for } |Y| < 1/2,$$

$$\phi_Y = 0 \qquad \text{on } Y = \pm 1/2.$$

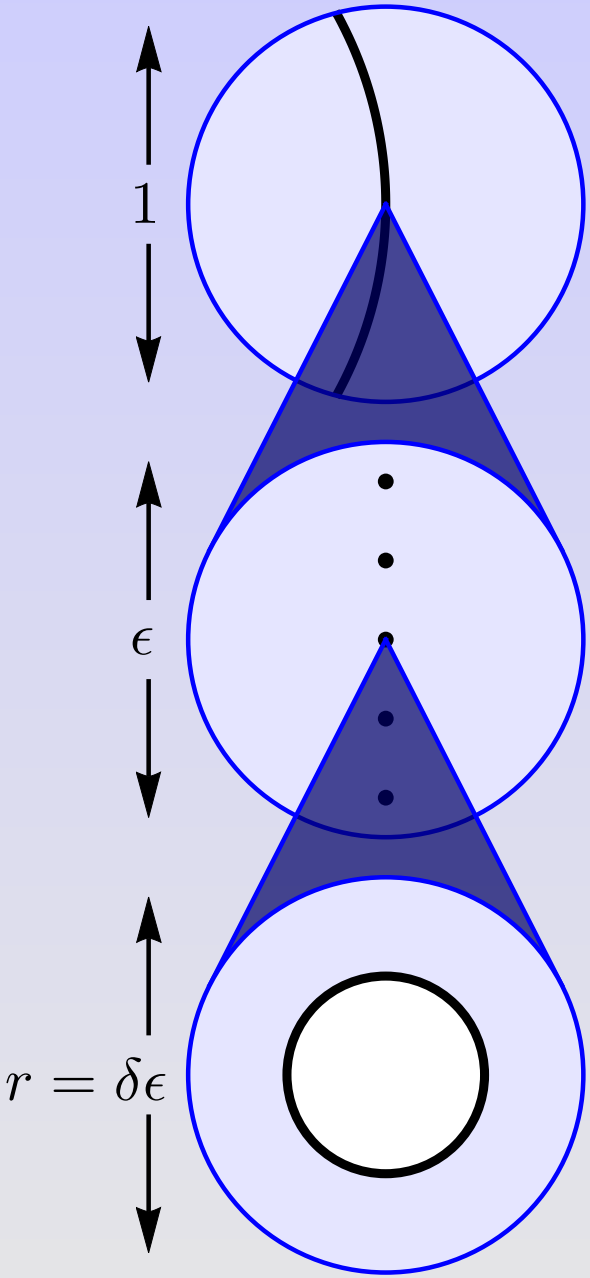
$C =$ charge on the wire.

$$w = B + C \log\left(\sinh\left(\pi Z\right)\right).$$

Inner region $(x, y) = (r\xi, r\eta)$.

$$w = \phi_0 + A \log \zeta.$$

Matching



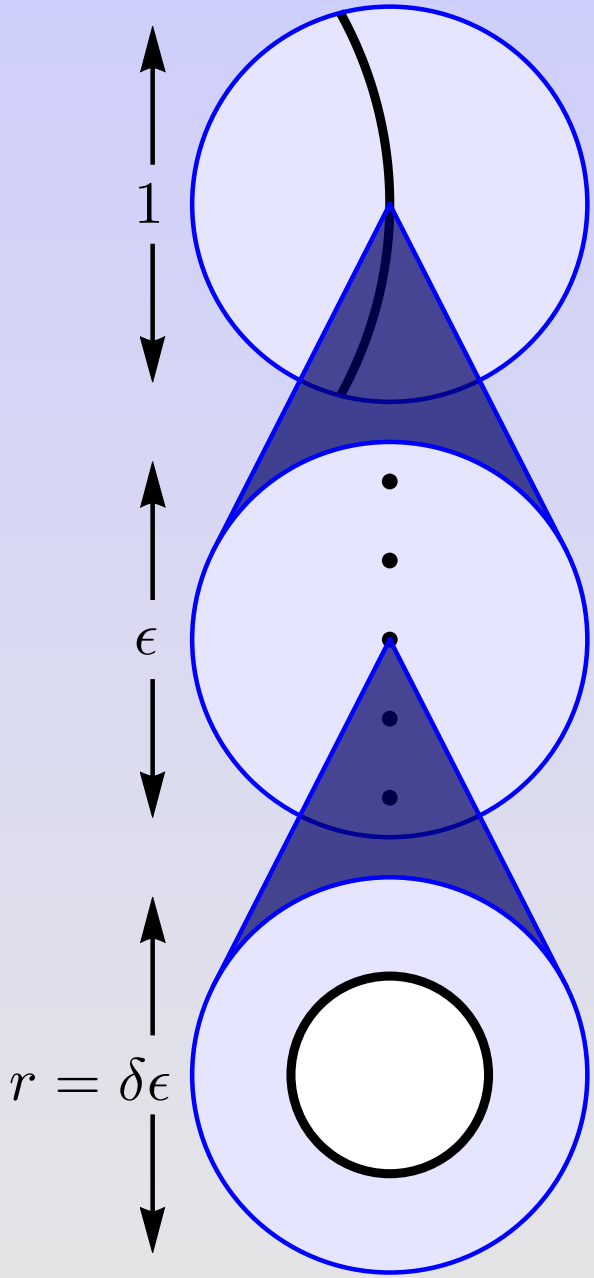
Middle region $(x, y) = (\epsilon X, \epsilon X)$.

$$w \sim B + C \log(\pi Z) \text{ as } Z \to 0.$$

Inner region $(x, y) = (r\xi, r\eta)$.

$$w \sim \phi_0 + A \log 1/\delta + A \log Z.$$

Matching



Matching gives

$$\phi_0 + A \log 1/\delta = B + C \log \pi,$$

$$A = C.$$

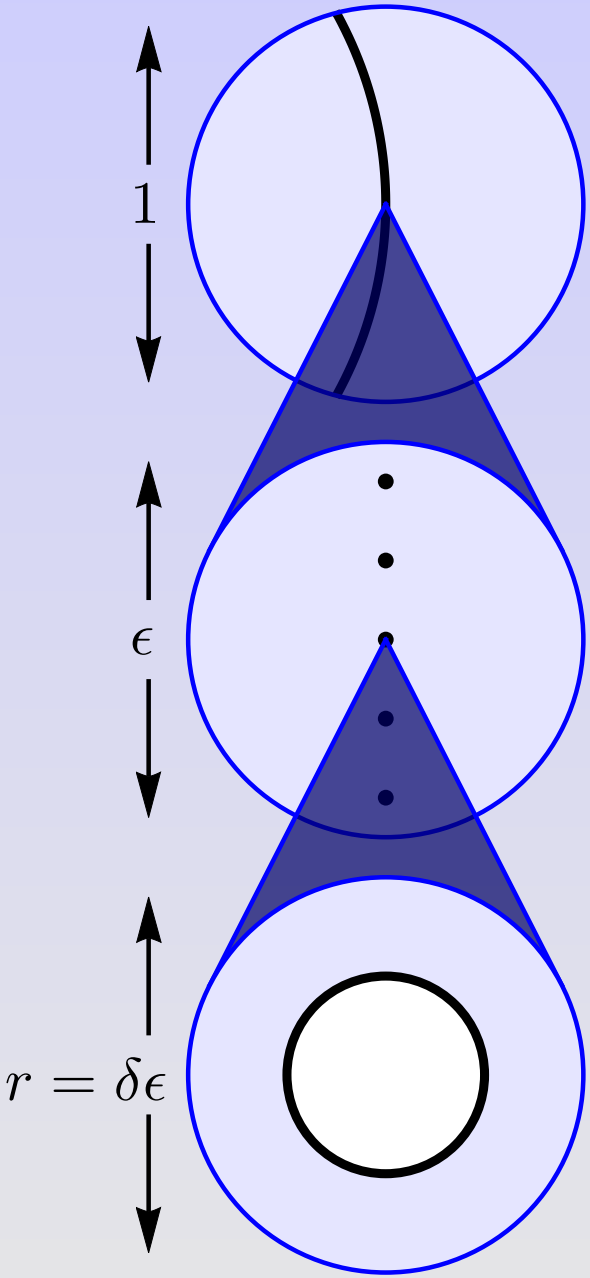
Middle region $(x, y) = (\epsilon X, \epsilon X)$.

$$w \sim B + C \log(\pi Z) \text{ as } Z \to 0.$$

Inner region $(x, y) = (r\xi, r\eta)$.

$$w \sim \phi_0 + A \log 1/\delta + A \log Z.$$

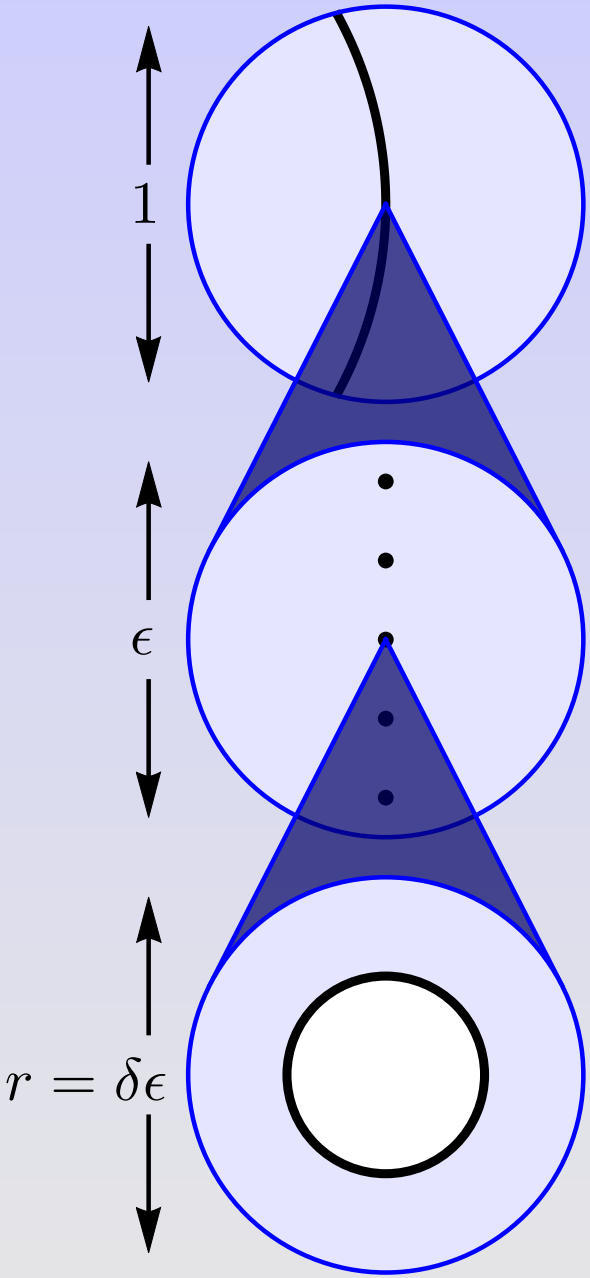
Matching



Middle region $(x, y) = (\epsilon X, \epsilon X)$.

$\phi \sim \pm C\pi X - C\log 2 + B$ as $X \to \pm\infty$.

Matching



Outer region

Matching gives

$$\left[\frac{\partial\phi}{\partial n}\right]_{\Gamma-}^{\Gamma+} = \frac{2\pi C}{\epsilon}, \quad \phi(\Gamma) = B - C\log 2.$$

Middle region $(x, y) = (\epsilon X, \epsilon X)$.

$\phi \sim \pm C\pi X - C\log 2 + B$ as $X \to \pm\infty$.

2. Multiple scales analysis

Eliminating A, B and C gives

$$\left[\frac{\partial\phi}{\partial n}\right]_-^+ = \alpha(\phi - \phi_0) \text{ on } \Gamma, \qquad \alpha = \frac{2\pi}{\epsilon\log(\epsilon/(2\pi r))} = \frac{n}{\log(1/(rn))}.$$

$\alpha(\phi - \phi_0) =$ charge density. $\alpha =$ capacitance per unit length.
Solution of homogenised problem has

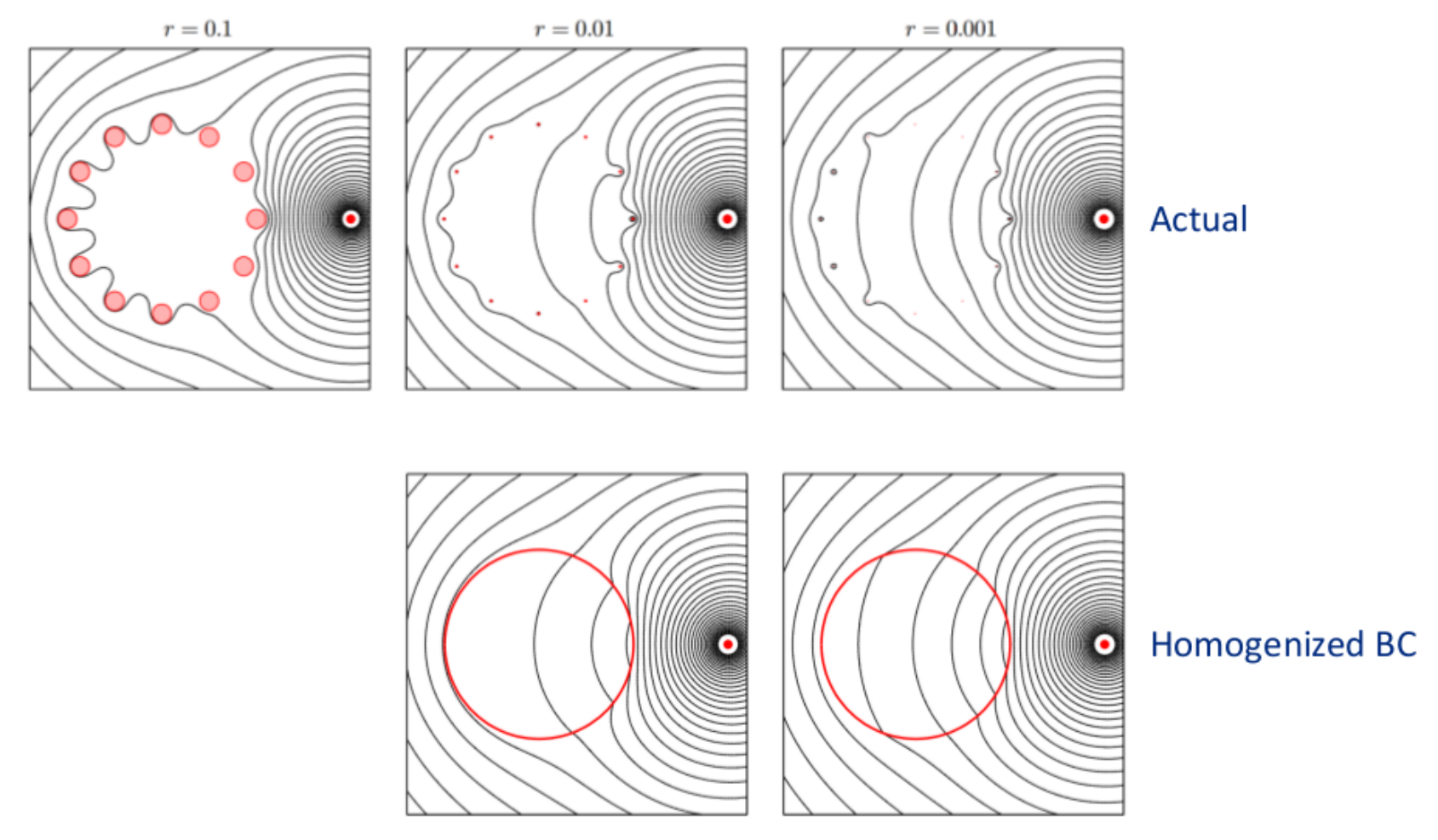
$$|\nabla\phi(0)| = \frac{2}{(2+\alpha)|z_s|}.$$

Effective screening requires $\alpha \gg 1$, i.e.

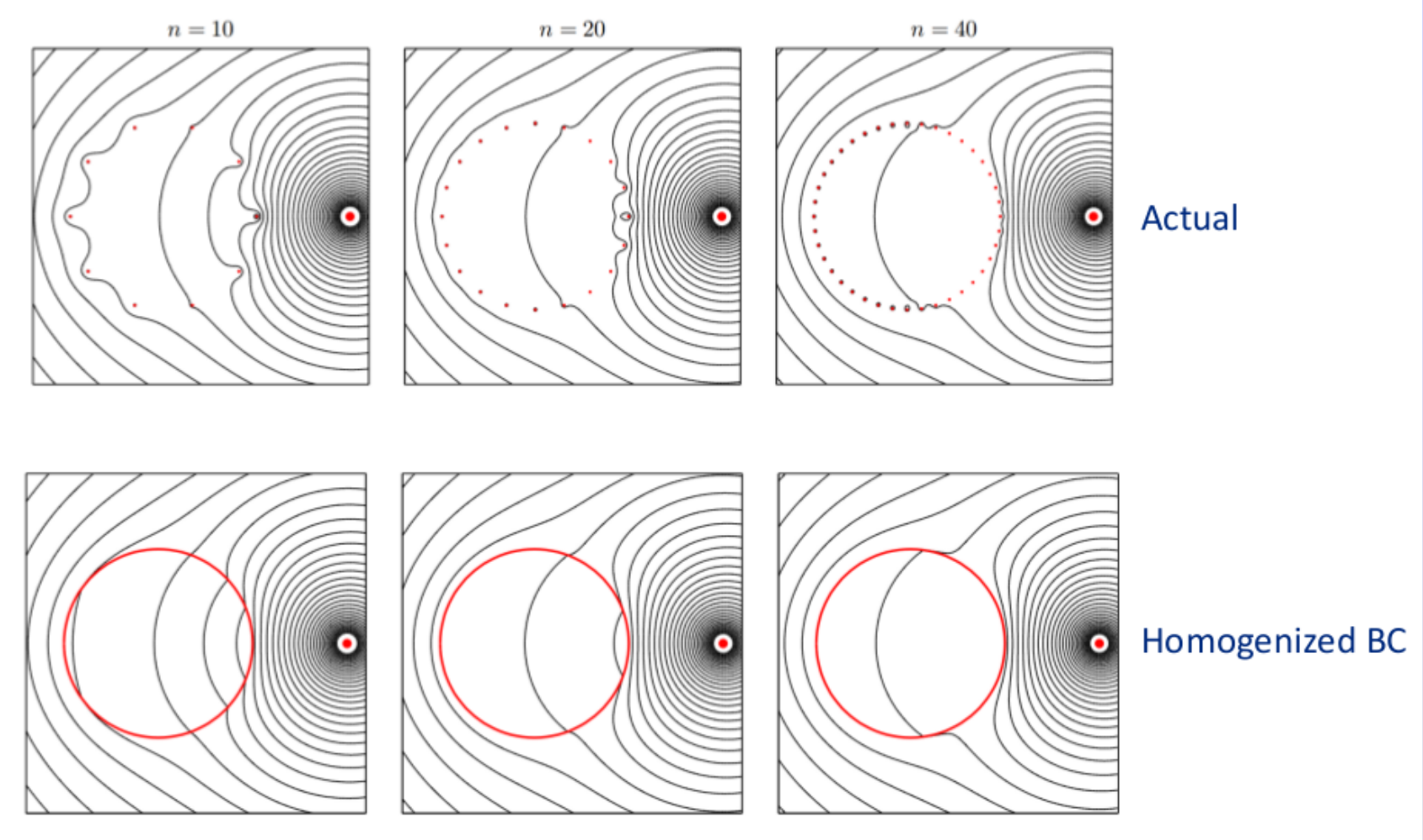
$$r \gg \frac{\mathrm{e}^{-n}}{n}.$$

Thus when there are many wires they can be very thin.

Dependence on r $(n = 12)$



Dependence on n $(r = 0.01)$



3. Point charges via constrained optimisation

A disk at z_k of radius $r \ll 1$ behaves like a point charge of some strength q_k. The challenge for Faraday cage analysis is, what is q_k? We can find these charges by energy minimisation.

Key observation: a charged disk has a self-energy of $-\frac{1}{2}q_k^2 \log r$. It takes work to force charge onto a small disk.

Find n charges q_k, summing to 0, that minimise the total energy:

$$E(\mathbf{q}) = -\frac{1}{2}\sum_{k=1}^{n} q_k^2 \log r - \sum_{k=1}^{n}\sum_{j>k} q_j q_k \log|z_k - z_j| - \sum_{k=1}^{n} q_k \log|z_k - z_s|$$

SELF-ENERGY OF DISKS

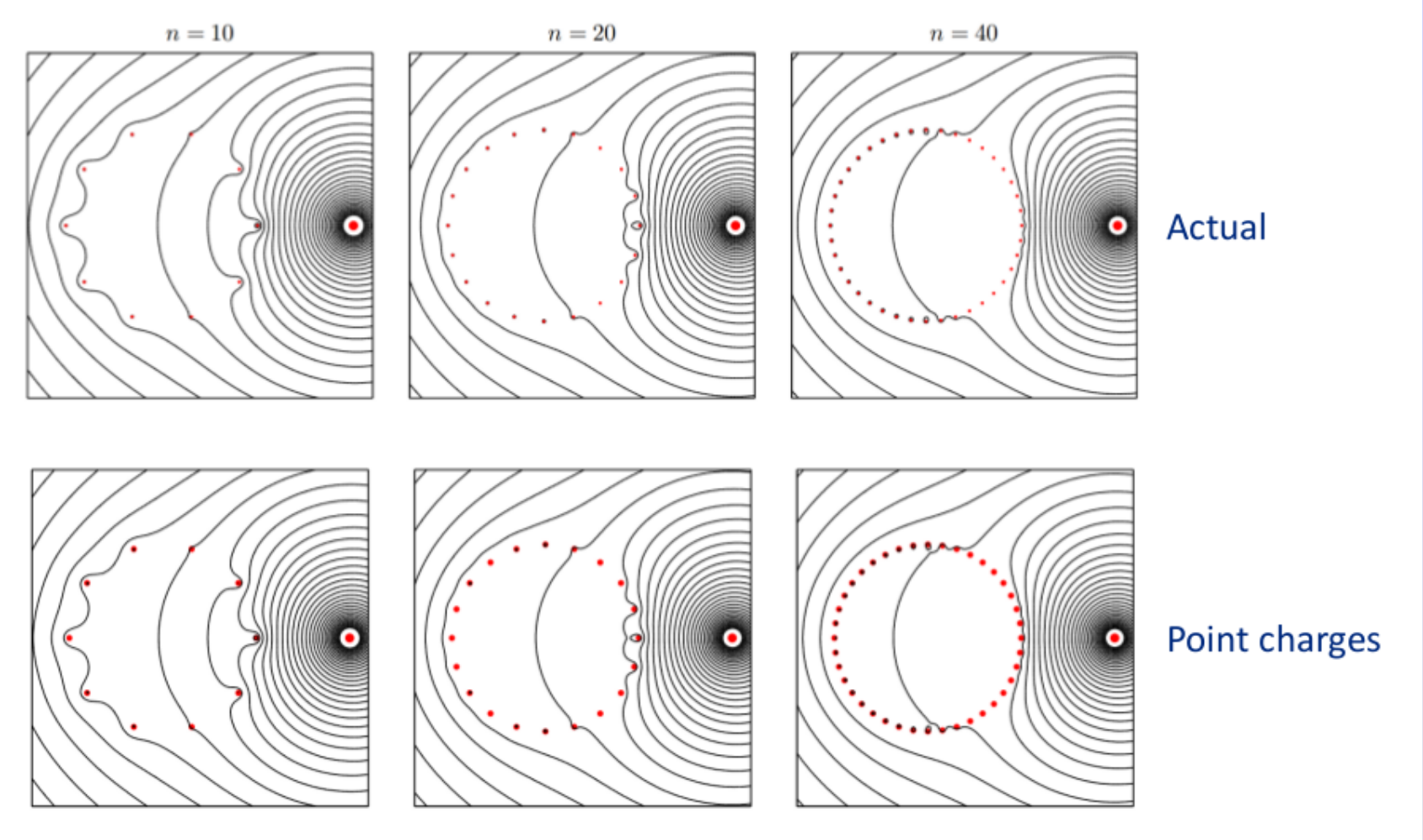
MUTUAL ENERGY BETWEEN DISKS

ENERGY OF DISKS IN EXTERNAL FIELD

This is a constrained quadratic programming problem

$$\text{minimise } E(\mathbf{q}) = \frac{1}{2}\mathbf{q}^T A\mathbf{q} - \mathbf{f}^T\mathbf{q}, \qquad \mathbf{c}^T\mathbf{q} = 0.$$

Dependence on n $(r = 0.01)$

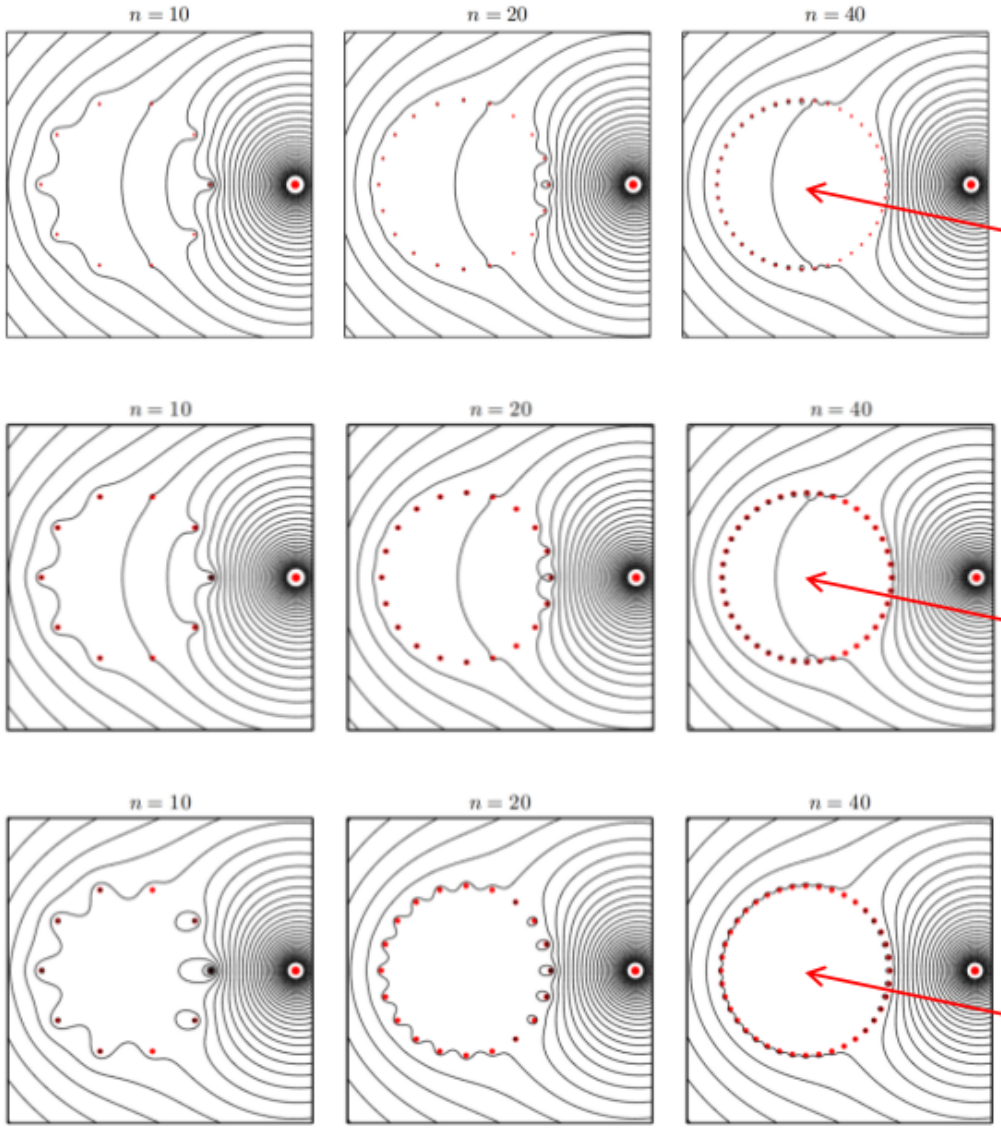


Back to trapezoidal rule

Small charged disks can be approximated by point charges to high accuracy.
A charge distribution on a smooth curve can be approximated by point samples to high accuracy—the trapezoidal rule.

So why doesn't the Faraday cage provide strong screening?

Because it provides an exponentially good approximation not to the zero field, but to the field associated with the homogenized boundary condition.



Finite wires, $r = 0.01$

field strength 0.023

Point charges from sampling solution from homogenized BC

field strength 0.022

Point charges from sampling solution from perfectly conducting shell

field strength 0.0000000000023