

Mathematical Challenges in Quantum Algorithms

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Introduction

There are many things we can do with our quantum computers. For example:

- Factorise large integers and hence **break RSA**;
- Efficiently **simulate** quantum-mechanical systems;
- Solve certain **search and optimisation** problems faster than possible classically;
- ...

See the [Quantum Algorithm Zoo](#)

(<http://math.nist.gov/quantum/zoo/>) for **214 219** papers on quantum algorithms.

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- What we **can't** do;
- **Why** we can do what we can.

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- What we **can't** do;
- **Why** we can do what we can.

In this talk I will discuss a personal selection of a few of these open problems. Notably, they (mostly) rest on purely **classical** mathematical questions!

Hidden subgroup problems

Hidden subgroup problem (e.g. [Boneh and Lipton '95])

Let G be a group. Given oracle access to a function $f : G \rightarrow X$ such that f is **constant** on the cosets of some subgroup $H \leq G$, and **distinct** on each coset, identify H .

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On a quantum computer, the HSP can be solved using $O(\log |G|)$ queries to f for all groups G [Ettinger et al. '04]. Classically, some groups require $\Omega(\sqrt{|G|})$ queries [Simon '97].

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The HSP is related to many other problems and cryptosystems:

Problem	Group	Complexity	Cryptosystem
Factorisation	\mathbb{Z}_N	Polynomial ¹	RSA
Discrete log	$\mathbb{Z}_{p-1} \times \mathbb{Z}_{p-1}$	Polynomial ¹	Diffie-Hellman, DSA, ...
Elliptic curve d. log	Elliptic curve	Polynomial ²	ECDH, ECDSA, ...
Principal ideal	\mathbb{R}	Polynomial ³	Buchmann-Williams
Shortest lattice vector	Dihedral grp	Subexp. ⁴	NTRU, Ajtai-Dwork, ...
Graph isomorphism	Symmetric grp	Exponential	—

¹Shor '97, ²Proos et al. '03, ³Hallgren '07, ⁴Kuperberg '05, Regev '04

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A significant amount of other work on the HSP has resolved its complexity for many other groups.

The dihedral hidden subgroup problem

The dihedral HSP turns out to be equivalent to a [hidden shift](#) problem:

- Given two injective functions $f, g : \mathbb{Z}_N \rightarrow X$ such that $g(x) = f(x + s)$ for some $s \in \mathbb{Z}_N$, determine s .



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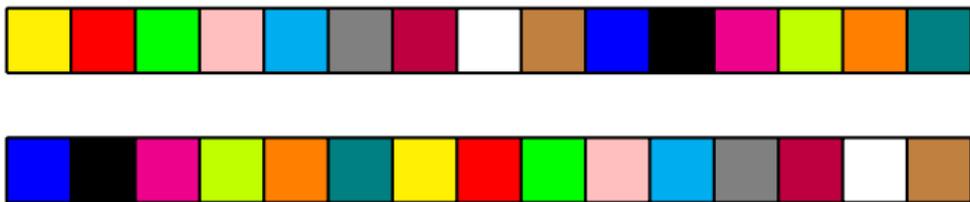


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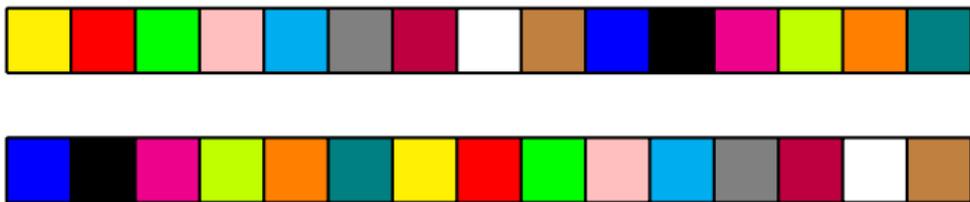
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- A $\text{poly}(\log N)$ -time algorithm would give an efficient quantum algorithm for the shortest vector problem in lattices [Regev '04].

Solving the HSP via the Kuperberg sieve

- One approach to solving the dihedral HSP starts by producing many quantum states of the form

$$|\psi_x\rangle := |0\rangle + e^{2\pi i s x/N} |1\rangle,$$

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- We would like to make the state $|\psi_{N/2}\rangle = |0\rangle + (-1)^s |1\rangle$, which is sufficient to determine one bit of s .
- One way to do this is via the following **combination** operation:

$$C(|\psi_x\rangle, |\psi_y\rangle) = \begin{cases} |\psi_{x+y}\rangle & \text{with prob. } 1/2 \\ |\psi_{x-y}\rangle & \text{with prob. } 1/2 \end{cases}$$

Solving the HSP via the Kuperberg sieve

Theorem [Kuperberg '05, AM '14]

It is sufficient to start with $2^{1.781\dots\sqrt{\log_2 N}}$ $\text{poly}(\log N)$ random states $|\psi_x\rangle$ to be able to produce a state of the form $|\psi_{N/2}\rangle$ with high probability using combination operations.

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Open problem

Can this be improved?

- Solving certain **average-case subset sum** problems efficiently would also give us an efficient solution to this problem [Regev '04].

Quantum query complexity

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Open problem

What is the largest possible separation between quantum and classical query complexity for a **total** function?

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What are these quantities?

- $\deg(f)$ is the degree of f as an n -variate polynomial over \mathbb{R} .
- $\widetilde{\deg}(f)$ is the **approximate** degree: i.e. the smallest degree of any polynomial \tilde{f} such that $|\tilde{f}(x) - f(x)| \leq 1/3$ for all x .

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For example: $\deg(\text{OR}_2) = 2$ $\text{OR}_2(x) = x_1 + x_2 - x_1x_2$
 $\widetilde{\deg}(\text{OR}_2) = 1$ e.g. $\widetilde{\text{OR}}_2(x) = (x_1 + x_2)/3$

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- $Q(f) = \Omega(\widetilde{\deg}(f))$ [Beals et al. '01];
- $Q(f) = \Omega(\sqrt{\text{bs}(f)})$ [Bennett et al. '97];
- $D(f) = O(\deg(f) \text{bs}(f))$ [Midrijānis '05].

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Hence we would have $D(f) \stackrel{?}{=} O(\widetilde{\deg}(f)^2 \text{bs}(f)) = O(Q(f)^4)$.

Quantum property testing

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Property testing

- Let X be a set of objects and $d : X \times X \rightarrow [0, 1]$ be a distance measure on X .
- A subset $\mathcal{P} \subseteq X$ is called a **property**.
- An object $x \in X$ is **ϵ -far** from \mathcal{P} if $d(x, y) \geq \epsilon$ for all $y \in \mathcal{P}$;
- x is **ϵ -close** to \mathcal{P} if there is a $y \in \mathcal{P}$ such that $d(x, y) \leq \epsilon$.

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An **ϵ -property tester** for \mathcal{P} is an algorithm that receives as input either an $x \in \mathcal{P}$ or an x that is ϵ -far from \mathcal{P} , and that distinguishes these two cases with success probability at least $2/3$.

Quantum property testing

In some cases, quantum property testers can significantly outperform their classical counterparts. For example:

- An exponential speedup for testing whether a sequence of N integers is **periodic** (i.e. $\text{poly}(\log N)$ vs. $\Omega(N^{1/4})$ queries) [Chakraborty et al. '10];
- Polynomial speedups for testing some properties of **graphs**: e.g. bipartiteness, expansion ($\tilde{O}(N^{1/3})$ vs. $\Omega(N^{1/2})$ queries in both cases) [Ambainis et al. '11];
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However, most known quantum property-testing algorithms are based around taking an **existing** quantum algorithm and adapting it for the property-testing setting.

Quantum property testing

Open problem

Could there be an exponential quantum speedup for testing a **graph property**?

- A graph property is simply a subset of the set of all adjacency matrices which is invariant under relabelling the graph vertices.
- Examples include bipartiteness, planarity, 3-colourability, connectivity ...
- No super-polynomial speedup is currently known.

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- So it seems that proving that an exponential speedup can, or cannot, exist would throw light on the **role of symmetry** in quantum algorithms.
- Also, classically the graph properties that are efficiently testable have been completely characterised [Alon et al. '09]. Can we use this characterisation quantumly?

Summary and further reading

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Some further reading:

- “Quantum algorithms for algebraic problems” [Childs and van Dam '08]
- “Quantum algorithms” [Mosca '08]
- “New developments in quantum algorithms” [Ambainis '10]
- “A survey of quantum property testing” [AM and de Wolf '13]

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Thanks!

Average-case simulation of quantum algorithms

Conjecture A [Aaronson and Ambainis '09, slightly modified]

Let Q be a quantum algorithm which makes T queries to x . Then, for any $\epsilon > 0$, there is a classical algorithm which makes $\text{poly}(T, 1/\epsilon, 1/\delta)$ queries to x , and approximates Q 's success probability to within $\pm\epsilon$ on a $1 - \delta$ fraction of inputs.

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- Given known results, essentially the **strongest conjecture** one could make about classical simulation of quantum query algorithms.
- Aaronson and Ambainis show that Conjecture A follows from the following, more mathematical conjecture...

Influences of variables on low-degree polynomials

Conjecture B [Aaronson and Ambainis '09]

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a degree d multivariate polynomial such that $0 \leq f(x) \leq 1$ for all $x \in \{\pm 1\}^n$ and $\text{Var}(f) \geq \epsilon$. Then there exists $j \in \{1, \dots, n\}$ such that

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In this conjecture:

$$\text{Var}(f) = \mathbb{E}[(f(x) - \mathbb{E}[f])^2] = \frac{1}{2^n} \sum_{x \in \{\pm 1\}^n} \left(f(x) - \frac{1}{2^n} \sum_{y \in \{\pm 1\}^n} f(y) \right)^2$$

$$\text{Inf}_j(f) = \frac{1}{2^{n+2}} \sum_{x \in \{\pm 1\}^n} (f(x) - f(x^j))^2$$

Influences of variables on low-degree polynomials

This conjecture has been proven in a couple of special cases:

- If f is symmetric under permutations of the input bits [Bačkurs '12];
- If f is a multilinear form whose coefficients are equal in absolute value [AM '12].

The general case remains open.