Inferring Interaction Rules from Observations of Evolutive Systems

(joint work with M. Fornasier, M. Hansen and M. Maggioni)

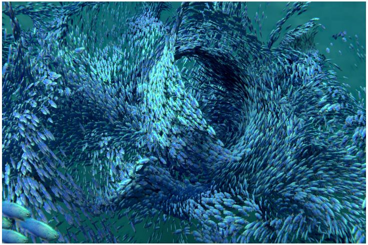
Mattia Bongini

Technische Universität München,
Department of Mathematics,
Chair of Applied Numerical Analysis

bongini@ma.tum.de

Data-Rich Phenomena
Cambridge
December 14-16, 2015

What is a self-organizing system?



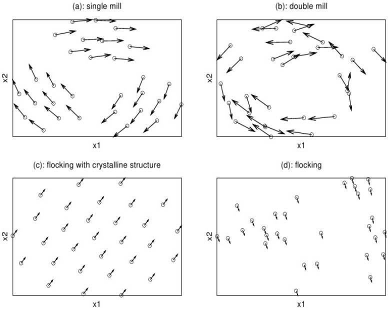
A framework for social dynamics

We consider multiagent systems of the form: for $i = 1, \dots, N$

$$\dot{x}_i = \frac{1}{N} \sum_{j=1}^{N} a(|x_i - x_j|)(x_j - x_i) \in \mathbb{R}^d,$$

and their mean-field limit equation (here $F[a](\xi) = -a(|\xi|)\xi$)

$$\frac{\partial \mu}{\partial t} = -\nabla \cdot ((F[a] * \mu)\mu),$$



Patterns related to different balances of social forces

Several "social forces" encoded in the interaction kernel a:

- alignment;
- repulsion-attraction;
- self-propulsion/friction...

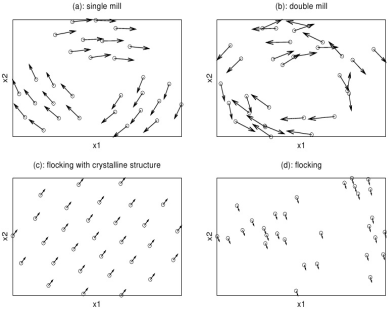
A framework for social dynamics

We consider multiagent systems of the form: for $i = 1, \dots, N$

$$\dot{x}_i = \frac{1}{N}\sum_{j=1}^{N} a(|x_i - x_j|)(x_j - x_i) \in \mathbb{R}^d,$$

and their mean-field limit equation (here $F[a](\xi) = -a(|\xi|)\xi$)

$$\frac{\partial \mu}{\partial t} = -\nabla \cdot ((F[a] * \mu)\mu),$$



Patterns related to different balances of social forces

Several "social forces" encoded in the interaction kernel a:

- alignment;
- repulsion-attraction;
- self-propulsion/friction...

The problem

- Tremendous theoretical success but the issue of actual applicability is so far scarcely addressed;
- Purely qualitative analysis to reproduce macroscopical patterns;
- Well-posedness relies on smoothness and asymptotic properties of the kernel a at 0 and ∞;
- Certainly results of great importance, as such functions likely differ from physical models: it is legitimate to consider a large variety of function classes;
- However, a solid mathematical framework on ‘learnability” of interactions from observations of the dynamics is not yet available.

The problem

- Tremendous theoretical success but the issue of actual applicability is so far scarcely addressed;
- Purely qualitative analysis to reproduce macroscopical patterns;
- Well-posedness relies on smoothness and asymptotic properties of the kernel a at 0 and ∞;
- Certainly results of great importance, as such functions likely differ from physical models: it is legitimate to consider a large variety of function classes;
- However, a solid mathematical framework on 'learnability" of interactions from observations of the dynamics is not yet available.

The problem

- Tremendous theoretical success but the issue of actual applicability is so far scarcely addressed;
- Purely qualitative analysis to reproduce macroscopical patterns;
- Well-posedness relies on smoothness and asymptotic properties of the kernel a at 0 and ∞;
- Certainly results of great importance, as such functions likely differ from physical models: it is legitimate to consider a large variety of function classes;
- However, a solid mathematical framework on 'learnability" of interactions from observations of the dynamics is not yet available.

The problem

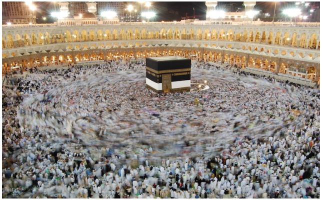
- Tremendous theoretical success but the issue of actual applicability is so far scarcely addressed;
- Purely qualitative analysis to reproduce macroscopical patterns;
- Well-posedness relies on smoothness and asymptotic properties of the kernel a at 0 and ∞;
- Certainly results of great importance, as such functions likely differ from physical models: it is legitimate to consider a large variety of function classes;
- However, a solid mathematical framework on ‘learnability” of interactions from observations of the dynamics is not yet available.

The problem

- Tremendous theoretical success but the issue of actual applicability is so far scarcely addressed;
- Purely qualitative analysis to reproduce macroscopical patterns;
- Well-posedness relies on smoothness and asymptotic properties of the kernel a at 0 and ∞;
- Certainly results of great importance, as such functions likely differ from physical models: it is legitimate to consider a large variety of function classes;
- However, a solid mathematical framework on ‘learnability” of interactions from observations of the dynamics is not yet available.

The problem

- Tremendous theoretical success but the issue of actual applicability is so far scarcely addressed;
- Purely qualitative analysis to reproduce macroscopical patterns;
- Well-posedness relies on smoothness and asymptotic properties of the kernel a at 0 and ∞;
- Certainly results of great importance, as such functions likely differ from physical models: it is legitimate to consider a large variety of function classes;
- However, a solid mathematical framework on 'learnability" of interactions from observations of the dynamics is not yet available.



$$\Rightarrow \quad a(r) = \sin\left(\log(r+1)\prod_{i=1}^{d} e^{\cos(i\cdot r^2)} + \dots\right)$$

The *naive* approach: optimal control

Given a finite time horizon $T > 0$ one would seek for a minimizer $\widehat{a}$ of

$$\mathcal{E}(\widehat{a}) = \frac{1}{T}\int_0^T \left[\|x[a](s) - x[\widehat{a}](s)\|^2 + \mathcal{R}(\widehat{a})\right] ds,$$

being $\mathcal{R}$ a suitable regularization functional and $t \mapsto x[\widehat{a}](t) = (x_1(t), \ldots, x_N(t))$ be the solution of

$$\dot{x}_i = \frac{1}{N}\sum_{j=1}^{N} \widehat{a}(|x_i - x_j|)(x_j - x_i).$$

However, one faces several problems!

- $t \mapsto x[\widehat{a}](t)$ strongly nonlinear $\Rightarrow \mathcal{E}(\widehat{a})$ strongly nonconvex;
- computationally unfeasible for N large (curse of dimensionality - Richard E. Bellman).

The *naive* approach: optimal control

Given a finite time horizon $T > 0$ one would seek for a minimizer $\widehat{a}$ of

$$\mathcal{E}(\widehat{a}) = \frac{1}{T}\int_0^T \left[\|x[a](s) - x[\widehat{a}](s)\|^2 + \mathcal{R}(\widehat{a})\right] ds,$$

being $\mathcal{R}$ a suitable regularization functional and $t \mapsto x[\widehat{a}](t) = (x_1(t), \ldots, x_N(t))$ be the solution of

$$\dot{x}_i = \frac{1}{N}\sum_{j=1}^{N} \widehat{a}(|x_i - x_j|)(x_j - x_i).$$

However, one faces several problems!

- $t \mapsto x[\widehat{a}](t)$ strongly nonlinear $\Rightarrow \mathcal{E}(\widehat{a})$ strongly nonconvex;
- computationally unfeasible for N large (curse of dimensionality - Richard E. Bellman).

The *naive* approach: optimal control

Given a finite time horizon $T > 0$ one would seek for a minimizer $\widehat{a}$ of

$$\mathcal{E}(\widehat{a}) = \frac{1}{T}\int_0^T \left[\|x[a](s) - x[\widehat{a}](s)\|^2 + \mathcal{R}(\widehat{a})\right] ds,$$

being $\mathcal{R}$ a suitable regularization functional and $t \mapsto x[\widehat{a}](t) = (x_1(t), \ldots, x_N(t))$ be the solution of

$$\dot{x}_i = \frac{1}{N}\sum_{j=1}^{N} \widehat{a}(|x_i - x_j|)(x_j - x_i).$$

However, one faces several problems!

- $t \mapsto x[\widehat{a}](t)$ strongly nonlinear $\Rightarrow \mathcal{E}(\widehat{a})$ strongly nonconvex;
- computationally unfeasible for N large (curse of dimensionality - Richard E. Bellman).

The *naive* approach: optimal control

Given a finite time horizon $T > 0$ one would seek for a minimizer $\widehat{a}$ of

$$\mathcal{E}(\widehat{a}) = \frac{1}{T}\int_0^T \left[\|x[a](s) - x[\widehat{a}](s)\|^2 + \mathcal{R}(\widehat{a})\right] ds,$$

being $\mathcal{R}$ a suitable regularization functional and $t \mapsto x[\widehat{a}](t) = (x_1(t), \ldots, x_N(t))$ be the solution of

$$\dot{x}_i = \frac{1}{N}\sum_{j=1}^{N} \widehat{a}(|x_i - x_j|)(x_j - x_i).$$

However, one faces several problems!

- $t \mapsto x[\widehat{a}](t)$ strongly nonlinear $\Rightarrow \mathcal{E}(\widehat{a})$ strongly nonconvex;
- computationally unfeasible for N large (curse of dimensionality - Richard E. Bellman).

The variational approach

Instead of minimizing the distances between trajectories $x[\widehat{a}](t)$, we minimize the discrepancy between velocities $\dot{x}_i(t)$,

$$\mathcal{E}_N(\widehat{a}) = \frac{1}{T}\int_0^T \frac{1}{N}\sum_{i=1}^N \left| \frac{1}{N}\sum_{j=1}^N \widehat{a}(|x_i(t) - x_j(t)|)(x_i(t) - x_j(t)) - \dot{x}_i(t) \right|^2 dt,$$

among all functions

$$\widehat{a} \in X = \{b : \mathbb{R}_+ \to \mathbb{R} \mid b \in L_\infty(\mathbb{R}_+) \cap W^1_{\infty,loc}(\mathbb{R}_+)\}$$

(ODEs and mean-field equations are well-posed).

Proposition

If $a, \widehat{a} \in X$ then there exist a constant $C > 0$ depending on $T, \widehat{a}$ and $x_{0,1}, \dots, x_{0,N}$ and a "certain" compact set $K \subset \mathbb{R}_+$ such that

$$\|x[a](t) - x[\widehat{a}](t)\| \leq C\sqrt{\mathcal{E}_N(\widehat{a})} \quad \textit{for all } t \in [0, T].$$

Hence, minimizing $\mathcal{E}_N(\widehat{a})$ implies an accurate approximation of $t \mapsto x[\widehat{a}](t)$ at finite time. (Proof: just a Gronwall.)

The variational approach

Instead of minimizing the distances between trajectories $x[\widehat{a}](t)$, we minimize the discrepancy between velocities $\dot{x}_i(t)$,

$$\mathcal{E}_N(\widehat{a}) = \frac{1}{T}\int_0^T \frac{1}{N}\sum_{i=1}^N \left| \frac{1}{N}\sum_{j=1}^N \widehat{a}(|x_i(t)-x_j(t)|)(x_i(t)-x_j(t)) - \dot{x}_i(t)\right|^2 dt,$$

among all functions

$$\widehat{a} \in X = \left\{ b : \mathbb{R}_+ \to \mathbb{R} \mid b \in L_\infty(\mathbb{R}_+) \cap W^1_{\infty,loc}(\mathbb{R}_+)\right\}$$

(ODEs and mean-field equations are well-posed).

Proposition

If $a, \widehat{a} \in X$ then there exist a constant $C > 0$ depending on $T, \widehat{a}$ and $x_{0,1}, \ldots, x_{0,N}$ and a "certain" compact set $K \subset \mathbb{R}_+$ such that

$$\|x[a](t) - x[\widehat{a}](t)\| \leq C\sqrt{\mathcal{E}_N(\widehat{a})} \qquad \text{for all } t \in [0,T].$$

Hence, minimizing $\mathcal{E}_N(\widehat{a})$ implies an accurate approximation of $t \mapsto x[\widehat{a}](t)$ at finite time. (Proof: just a Gronwall.)

The variational approach

Instead of minimizing the distances between trajectories $x[\widehat{a}](t)$, we minimize the discrepancy between velocities $\dot{x}_i(t)$,

$$\mathcal{E}_N(\widehat{a}) = \frac{1}{T}\int_0^T \frac{1}{N}\sum_{i=1}^N \left| \frac{1}{N}\sum_{j=1}^N \widehat{a}(|x_i(t)-x_j(t)|)(x_i(t)-x_j(t)) - \dot{x}_i(t) \right|^2 dt,$$

among all functions

$$\widehat{a} \in X = \left\{ b : \mathbb{R}_+ \to \mathbb{R} \mid b \in L_\infty(\mathbb{R}_+) \cap W^1_{\infty,loc}(\mathbb{R}_+) \right\}$$

(ODEs and mean-field equations are well-posed).

Proposition

If $a, \widehat{a} \in X$ then there exist a constant $C > 0$ depending on $T, \widehat{a}$ and $x_{0,1}, \ldots, x_{0,N}$ and a "certain" compact set $K \subset \mathbb{R}_+$ such that

$$\|x[a](t) - x[\widehat{a}](t)\| \leq C\sqrt{\mathcal{E}_N(\widehat{a})} \quad \textit{for all } t \in [0,T].$$

Hence, minimizing $\mathcal{E}_N(\widehat{a})$ implies an accurate approximation of $t \mapsto x[\widehat{a}](t)$ at finite time. (Proof: just a Gronwall.)

The main question

- $\mathcal{E}_N$ is easily computable from the knowledge of the trajectories of the system $x[a](t)$ (perhaps approximating $\dot{x}_i(t)$ by $\frac{x_i(t+\delta t)-x_i(t)}{\delta t}$).
- We use the number of agents N as an optimization parameter: does a larger number of agents improve learnability?
- Being quadratic (its minimization is just a least squares!!!), its minimizers can be efficiently numerically computed on a finite dimensional space $V_N \subset X$ such that $V_N \nearrow X$ as $N \to +\infty$.

Question: for which sequence V_N do minimizers

$$\widehat{a}_N \in \operatorname*{argmin}_{\widehat{a} \in V_N} \mathcal{E}_N(\widehat{a})$$

satisfy $\widehat{a}_N \to a$, and in which topology does this limit hold?

The main question

- $\mathcal{E}_N$ is easily computable from the knowledge of the trajectories of the system $x[a](t)$ (perhaps approximating $\dot{x}_i(t)$ by $\frac{x_i(t+\delta t)-x_i(t)}{\delta t}$).
- We use the number of agents N as an optimization parameter: does a larger number of agents improve learnability?
- Being quadratic (its minimization is just a least squares!!!), its minimizers can be efficiently numerically computed on a finite dimensional space $V_N \subset X$ such that $V_N \nearrow X$ as $N \to +\infty$.

Question: for which sequence V_N do minimizers

$$\widehat{a}_N \in \operatorname*{argmin}_{\widehat{a} \in V_N} \mathcal{E}_N(\widehat{a})$$

satisfy $\widehat{a}_N \to a$, and in which topology does this limit hold?

The main question

- $\mathcal{E}_N$ is easily computable from the knowledge of the trajectories of the system $x[a](t)$ (perhaps approximating $\dot{x}_i(t)$ by $\frac{x_i(t+\delta t)-x_i(t)}{\delta t}$).
- We use the number of agents N as an optimization parameter: does a larger number of agents improve learnability?
- Being quadratic (its minimization is just a least squares!!!), its minimizers can be efficiently numerically computed on a finite dimensional space $V_N \subset X$ such that $V_N \nearrow X$ as $N \to +\infty$.

Question: for which sequence V_N do minimizers

$$\widehat{a}_N \in \operatorname*{argmin}_{\widehat{a} \in V_N} \mathcal{E}_N(\widehat{a})$$

satisfy $\widehat{a}_N \to a$, and in which topology does this limit hold?

The main question

- $\mathcal{E}_N$ is easily computable from the knowledge of the trajectories of the system $x[a](t)$ (perhaps approximating $\dot{x}_i(t)$ by $\frac{x_i(t+\delta t)-x_i(t)}{\delta t}$).
- We use the number of agents N as an optimization parameter: does a larger number of agents improve learnability?
- Being quadratic (its minimization is just a least squares!!!), its minimizers can be efficiently numerically computed on a finite dimensional space $V_N \subset X$ such that $V_N \nearrow X$ as $N \to +\infty$.

Question: for which sequence V_N do minimizers

$$\widehat{a}_N \in \operatorname*{argmin}_{\widehat{a} \in V_N} \mathcal{E}_N(\widehat{a})$$

satisfy $\widehat{a}_N \to a$, and in which topology does this limit hold?

Γ-limit of the $\mathcal{E}_N$

- Suppose there exists a functional $\mathcal{E}$ such that $a = \operatorname{argmin}_{\widehat{a} \in X} \mathcal{E}(\widehat{a})$.
- Then the above question translates into the convergence of the minimizers of $\mathcal{E}_N$ to the minimizer of $\mathcal{E}$, i.e., the Γ-convergence of $\mathcal{E}_N$ to $\mathcal{E}$. But what can $\mathcal{E}$ be?
- Set $F[a](\xi) = -a(|\xi|)\xi$ and rewrite the initial system as

$$\begin{cases} \dot{x}_i^N(t) = \frac{1}{N} \sum_{j=1}^{N} F[a](x_i^N(t) - x_j^N(t)) \text{ for } t \in (0,T], \\ x_i^N(0) = x_{0,i}^N, \end{cases} \quad i = 1, \ldots, N.$$

- Define the empirical measure $\mu^N : [0,T] \to \mathcal{P}_c(\mathbb{R}^d)$ as

$$\mu^N(t) = \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i^N(t)}.$$

Γ-limit of the $\mathcal{E}_N$

- Suppose there exists a functional $\mathcal{E}$ such that $a = \operatorname{argmin}_{\widehat{a} \in X} \mathcal{E}(\widehat{a})$.
- Then the above question translates into the convergence of the minimizers of $\mathcal{E}_N$ to the minimizer of $\mathcal{E}$, i.e., the Γ-convergence of $\mathcal{E}_N$ to $\mathcal{E}$. But what can $\mathcal{E}$ be?
- Set $F[a](\xi) = -a(|\xi|)\xi$ and rewrite the initial system as

$$\begin{cases} \dot{x}_i^N(t) = \frac{1}{N}\sum_{j=1}^{N} F[a](x_i^N(t) - x_j^N(t)) \quad \text{for } t \in (0,T], \\ x_i^N(0) = x_{0,i}^N, \end{cases} \quad i = 1, \dots, N.$$

- Define the empirical measure $\mu^N : [0,T] \to \mathcal{P}_c(\mathbb{R}^d)$ as

$$\mu^N(t) = \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i^N(t)}.$$

Γ-limit of the $\mathcal{E}_N$

- Suppose there exists a functional $\mathcal{E}$ such that $a = \operatorname{argmin}_{\widehat{a} \in X} \mathcal{E}(\widehat{a})$.
- Then the above question translates into the convergence of the minimizers of $\mathcal{E}_N$ to the minimizer of $\mathcal{E}$, i.e., the Γ-convergence of $\mathcal{E}_N$ to $\mathcal{E}$. But what can $\mathcal{E}$ be?
- Set $F[a](\xi) = -a(|\xi|)\xi$ and rewrite the initial system as

$$\begin{cases} \dot{x}_i^N(t) = \frac{1}{N}\sum_{j=1}^N F[a](x_i^N(t) - x_j^N(t)) \ \text{ for } t \in (0,T], \\ x_i^N(0) = x_{0,i}^N, \end{cases} \quad i = 1, \ldots, N.$$

- Define the empirical measure $\mu^N : [0,T] \to \mathcal{P}_c(\mathbb{R}^d)$ as

$$\mu^N(t) = \frac{1}{N}\sum_{i=1}^N \delta_{x_i^N(t)}.$$

Γ-limit of the $\mathcal{E}_N$

- Suppose there exists a functional $\mathcal{E}$ such that $a = \operatorname{argmin}_{\widehat{a}\in X} \mathcal{E}(\widehat{a})$.
- Then the above question translates into the convergence of the minimizers of $\mathcal{E}_N$ to the minimizer of $\mathcal{E}$, i.e., the Γ-convergence of $\mathcal{E}_N$ to $\mathcal{E}$. But what can $\mathcal{E}$ be?
- Set $F[a](\xi) = -a(|\xi|)\xi$ and rewrite the initial system as

$$\begin{cases} \dot{x}_i^N(t) = \frac{1}{N}\sum_{j=1}^N F[a](x_i^N(t) - x_j^N(t)) \quad \text{for } t \in (0,T], \\ x_i^N(0) = x_{0,i}^N, \end{cases} \quad i = 1,\ldots,N.$$

- Define the empirical measure $\mu^N : [0,T] \to \mathcal{P}_c(\mathbb{R}^d)$ as

$$\mu^N(t) = \frac{1}{N}\sum_{i=1}^N \delta_{x_i^N(t)}.$$

Γ-limit of the $\mathcal{E}_N$

- Suppose there exists a functional $\mathcal{E}$ such that $a = \operatorname{argmin}_{\widehat{a} \in X} \mathcal{E}(\widehat{a})$.
- Then the above question translates into the convergence of the minimizers of $\mathcal{E}_N$ to the minimizer of $\mathcal{E}$, i.e., the Γ-convergence of $\mathcal{E}_N$ to $\mathcal{E}$. But what can $\mathcal{E}$ be?
- Set $F[a](\xi) = -a(|\xi|)\xi$ and rewrite the initial system as

$$\begin{cases} \dot{x}_i^N(t) = \dfrac{1}{N} \displaystyle\sum_{j=1}^{N} F[a](x_i^N(t) - x_j^N(t)) \quad \text{for } t \in (0, T], \\ x_i^N(0) = x_{0,i}^N, \end{cases} \quad i = 1, \ldots, N.$$

- Define the empirical measure $\mu^N : [0, T] \to \mathcal{P}_c(\mathbb{R}^d)$ as

$$\mu^N(t) = \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i^N(t)}.$$

A possible solution: the continuity equation

- μ^N is a solution to the continuity equation (abbreviated c.e.)

$$\frac{\partial \mu}{\partial t}(t) = -\nabla \cdot ((F[a] * \mu(t))\mu(t)) \quad \text{for } t \in (0, T].$$

with initial datum $\mu^N(0) = \mu_0^N := \frac{1}{N}\sum_{i=1}^N \delta_{x_{0,i}^N}$.

- Moreover, we can now rewrite $\mathcal{E}_N$ as

$$\begin{aligned}\mathcal{E}_N(\widehat{a}) &= \frac{1}{T}\int_0^T \frac{1}{N}\sum_{i=1}^N \left| \frac{1}{N}\sum_{j=1}^N (F[\widehat{a}] - F[a])(x_i - x_j)\right|^2 dt \\ &= \frac{1}{T}\int_0^T \int_{\mathbb{R}^d} \left| (F[\widehat{a}] - F[a]) * \mu^N(t)\right|^2 d\mu^N(t)(x)dt,\end{aligned}$$

A possible solution: the continuity equation

- μ^N is a solution to the continuity equation (abbreviated c.e.)

$$\frac{\partial \mu}{\partial t}(t) = -\nabla \cdot ((F[a] * \mu(t))\mu(t)) \quad \text{for } t \in (0, T].$$

with initial datum $\mu^N(0) = \mu_0^N := \frac{1}{N}\sum_{i=1}^N \delta_{x_{0,i}^N}$.

- Moreover, we can now rewrite $\mathcal{E}_N$ as

$$\mathcal{E}_N(\widehat{a}) = \frac{1}{T}\int_0^T \frac{1}{N}\sum_{i=1}^N \left| \frac{1}{N}\sum_{j=1}^N \big(F[\widehat{a}] - F[a]\big)(x_i - x_j)\right|^2 dt$$

$$= \frac{1}{T}\int_0^T \int_{\mathbb{R}^d} \left| \big(F[\widehat{a}] - F[a]\big) * \mu^N(t)\right|^2 d\mu^N(t)(x) dt,$$

A possible solution: the continuity equation

- μ^N is a solution to the continuity equation (abbreviated c.e.)

$$\frac{\partial \mu}{\partial t}(t) = -\nabla \cdot ((F[a] * \mu(t))\mu(t)) \quad \text{for } t \in (0, T].$$

with initial datum $\mu^N(0) = \mu_0^N := \frac{1}{N}\sum_{i=1}^N \delta_{x_{0,i}^N}$.

- Moreover, we can now rewrite $\mathcal{E}_N$ as

$$\begin{aligned}\mathcal{E}_N(\widehat{a}) &= \frac{1}{T}\int_0^T \frac{1}{N}\sum_{i=1}^N \left| \frac{1}{N}\sum_{j=1}^N \left(F[\widehat{a}] - F[a]\right)(x_i - x_j)\right|^2 dt \\ &= \frac{1}{T}\int_0^T \int_{\mathbb{R}^d} \left|\left(F[\widehat{a}] - F[a]\right) * \mu^N(t)\right|^2 d\mu^N(t)(x) dt,\end{aligned}$$

Existence and uniqueness of solutions of the c.e.

Theorem

- *Suppose* $a \in X$, *let* $T > 0$ *and fix* $\mu_0 \in \mathcal{P}_c(\mathbb{R}^d)$.
- *Let* μ *be a weak solution of the c.e. with* $\mu(0) = \mu_0$ *on* $[0, T]$.
- *Let* $\mu_0^N = \frac{1}{N}\sum_{i=1}^N \delta_{x_{0,i}^N}$ *be such that* $x_{0,i}^N \sim \mu_0$ *i.i.d.* $\forall N$ *and* $\forall i$.

Then, $\exists R > 0$ *depending only on* $T, a,$ *and* $\mathrm{supp}(\mu_0)$ *such that it holds*

$$\mathrm{supp}(\mu^N(t)) \cup \mathrm{supp}(\mu(t)) \subseteq B(0, R), \forall N \in \mathbb{N} \text{ and } \forall t \in [0, T],$$

$$\lim_{N \to +\infty} \sup_{t \in [0,T]} \mathcal{W}_1(\mu(t), \mu^N(t)) = 0.$$

Existence and uniqueness of solutions of the c.e.

Theorem

- *Suppose* $a \in X$, *let* $T > 0$ *and fix* $\mu_0 \in \mathcal{P}_c(\mathbb{R}^d)$.
- *Let* μ *be a weak solution of the c.e. with* $\mu(0) = \mu_0$ *on* $[0, T]$.
- *Let* $\mu_0^N = \frac{1}{N} \sum_{i=1}^N \delta_{x_{0,i}^N}$ *be such that* $x_{0,i}^N \sim \mu_0$ *i.i.d.* $\forall N$ *and* $\forall i$.

Then, $\exists R > 0$ *depending only on* $T, a,$ *and* $\operatorname{supp}(\mu_0)$ *such that it holds*

$$\operatorname{supp}(\mu^N(t)) \cup \operatorname{supp}(\mu(t)) \subseteq B(0, R), \forall N \in \mathbb{N} \text{ and } \forall t \in [0, T],$$

$$\lim_{N \to +\infty} \sup_{t \in [0,T]} \mathcal{W}_1(\mu(t), \mu^N(t)) = 0.$$

Existence and uniqueness of solutions of the c.e.

Theorem

- *Suppose* $a \in X$, *let* $T > 0$ *and fix* $\mu_0 \in \mathcal{P}_c(\mathbb{R}^d)$.
- *Let* μ *be a weak solution of the c.e. with* $\mu(0) = \mu_0$ *on* $[0, T]$.
- *Let* $\mu_0^N = \frac{1}{N}\sum_{i=1}^N \delta_{x_{0,i}^N}$ *be such that* $x_{0,i}^N \sim \mu_0$ *i.i.d.* $\forall N$ *and* $\forall i$.

Then, $\exists R > 0$ *depending only on* $T, a,$ *and* $\operatorname{supp}(\mu_0)$ *such that it holds*

$$\operatorname{supp}(\mu^N(t)) \cup \operatorname{supp}(\mu(t)) \subseteq B(0, R), \forall N \in \mathbb{N} \text{ and } \forall t \in [0, T],$$

$$\lim_{N \to +\infty} \sup_{t \in [0,T]} \mathcal{W}_1(\mu(t), \mu^N(t)) = 0.$$

Existence and uniqueness of solutions of the c.e.

Theorem

- *Suppose* $a \in X$, *let* $T > 0$ *and fix* $\mu_0 \in \mathcal{P}_c(\mathbb{R}^d)$.
- *Let* μ *be a weak solution of the c.e. with* $\mu(0) = \mu_0$ *on* $[0, T]$.
- *Let* $\mu_0^N = \frac{1}{N}\sum_{i=1}^N \delta_{x_{0,i}^N}$ *be such that* $x_{0,i}^N \sim \mu_0$ *i.i.d.* $\forall N$ *and* $\forall i$.

Then, $\exists R > 0$ *depending only on* $T, a,$ *and* $\operatorname{supp}(\mu_0)$ *such that it holds*

$$\operatorname{supp}(\mu^N(t)) \cup \operatorname{supp}(\mu(t)) \subseteq B(0,R), \forall N \in \mathbb{N} \text{ and } \forall t \in [0,T],$$

$$\lim_{N\to+\infty} \sup_{t\in[0,T]} \mathcal{W}_1(\mu(t),\mu^N(t)) = 0.$$

Existence and uniqueness of solutions of the c.e.

Theorem

- *Suppose* $a \in X$, *let* $T > 0$ *and fix* $\mu_0 \in \mathcal{P}_c(\mathbb{R}^d)$.
- *Let* μ *be a weak solution of the c.e. with* $\mu(0) = \mu_0$ *on* $[0, T]$.
- *Let* $\mu_0^N = \frac{1}{N}\sum_{i=1}^N \delta_{x_{0,i}^N}$ *be such that* $x_{0,i}^N \sim \mu_0$ *i.i.d.* $\forall N$ *and* $\forall i$.

Then, $\exists R > 0$ *depending only on* $T, a,$ *and* $\mathrm{supp}(\mu_0)$ *such that it holds*

$$\mathrm{supp}(\mu^N(t)) \cup \mathrm{supp}(\mu(t)) \subseteq B(0, R), \forall N \in \mathbb{N} \textit{ and } \forall t \in [0, T],$$

$$\lim_{N \to +\infty} \sup_{t \in [0,T]} \mathcal{W}_1(\mu(t), \mu^N(t)) = 0.$$

Existence and uniqueness of solutions of the c.e.

Theorem

- *Suppose* $a \in X$, *let* $T > 0$ *and fix* $\mu_0 \in \mathcal{P}_c(\mathbb{R}^d)$.
- *Let* μ *be a weak solution of the c.e. with* $\mu(0) = \mu_0$ *on* $[0, T]$.
- *Let* $\mu_0^N = \frac{1}{N}\sum_{i=1}^N \delta_{x_{0,i}^N}$ *be such that* $x_{0,i}^N \sim \mu_0$ *i.i.d.* $\forall N$ *and* $\forall i$.

Then, $\exists R > 0$ *depending only on* $T, a,$ *and* $\operatorname{supp}(\mu_0)$ *such that it holds*

$$\operatorname{supp}(\mu^N(t)) \cup \operatorname{supp}(\mu(t)) \subseteq B(0, R), \forall N \in \mathbb{N} \textit{ and } \forall t \in [0, T],$$

$$\lim_{N\to+\infty} \sup_{t\in[0,T]} \mathcal{W}_1(\mu(t), \mu^N(t)) = 0.$$

The limit functional $\mathcal{E}$

- Natural candidate for the Γ-limit $\mathcal{E}$ of the $\mathcal{E}_N$: as μ is the uniform limit of the μ^N then we define

$$\mathcal{E}(\widehat{a}) = \frac{1}{T}\int_0^T \int_{\mathbb{R}^d} \left| (F[\widehat{a}] - F[a]) * \mu(t) \right|^2 d\mu(t)(x)dt.$$

- Since $\mathcal{E}(\widehat{a}) \geq 0$ and $\mathcal{E}(a) = 0$, then a minimizes $\mathcal{E}$. Is it unique?
- Given $d(x,y) = |x-y|$, introduce the family of measures

$$\varrho(t)(A) = (\mu(t) \otimes \mu(t))(d^{-1}(A)), \text{ for all } t \in [0,T] \text{ and}$$

$$\rho(A) = \frac{1}{T}\int_0^T \int_A s^2 d\varrho(t)(s)dt$$

$\varrho(t)(s)$ tells precisely how likely is that there are two indexes $i \neq j$ such that $|x_i(t) - x_j(t)| = s$, while $\rho(s)$ averages these likelihoods over the entire time frame $[0,T]$ (weighted by s^2).

- $\mathrm{supp}(\rho) \subseteq [0, 2R]$.

The limit functional $\mathcal{E}$

- Natural candidate for the Γ-limit $\mathcal{E}$ of the $\mathcal{E}_N$: as μ is the uniform limit of the μ^N then we define

$$\mathcal{E}(\widehat{a}) = \frac{1}{T}\int_0^T \int_{\mathbb{R}^d} \left|\left(F[\widehat{a}] - F[a]\right) * \mu(t)\right|^2 d\mu(t)(x)dt.$$

- Since $\mathcal{E}(\widehat{a}) \geq 0$ and $\mathcal{E}(a) = 0$, then a minimizes $\mathcal{E}$. Is it unique?
- Given $d(x,y) = |x-y|$, introduce the family of measures

$$\varrho(t)(A) = (\mu(t) \otimes \mu(t))(d^{-1}(A)), \text{ for all } t \in [0,T] \text{ and}$$

$$\rho(A) = \frac{1}{T}\int_0^T \int_A s^2 d\varrho(t)(s)dt$$

$\varrho(t)(s)$ tells precisely how likely is that there are two indexes $i \neq j$ such that $|x_i(t) - x_j(t)| = s$, while $\rho(s)$ averages these likelihoods over the entire time frame $[0,T]$ (weighted by s^2).

- $\operatorname{supp}(\rho) \subseteq [0, 2R]$.

The limit functional $\mathcal{E}$

- Natural candidate for the Γ-limit $\mathcal{E}$ of the $\mathcal{E}_N$: as μ is the uniform limit of the μ^N then we define

$$\mathcal{E}(\widehat{a}) = \frac{1}{T}\int_0^T \int_{\mathbb{R}^d} \left| \left(F[\widehat{a}] - F[a]\right) * \mu(t) \right|^2 d\mu(t)(x)dt.$$

- Since $\mathcal{E}(\widehat{a}) \geq 0$ and $\mathcal{E}(a) = 0$, then a minimizes $\mathcal{E}$. Is it unique?
- Given $d(x,y) = |x-y|$, introduce the family of measures

$$\varrho(t)(A) = (\mu(t) \otimes \mu(t))(d^{-1}(A)), \text{ for all } t \in [0,T] \text{ and}$$

$$\rho(A) = \frac{1}{T}\int_0^T \int_A s^2 d\varrho(t)(s)dt$$

$\varrho(t)(s)$ tells precisely how likely is that there are two indexes $i \neq j$ such that $|x_i(t) - x_j(t)| = s$, while $\rho(s)$ averages these likelihoods over the entire time frame $[0,T]$ (weighted by s^2).
- $\text{supp}(\rho) \subseteq [0, 2R]$.

The limit functional $\mathcal{E}$

- Natural candidate for the Γ-limit $\mathcal{E}$ of the $\mathcal{E}_N$: as μ is the uniform limit of the μ^N then we define

$$\mathcal{E}(\widehat{a}) = \frac{1}{T}\int_0^T \int_{\mathbb{R}^d} \left| \left(F[\widehat{a}] - F[a]\right) * \mu(t) \right|^2 d\mu(t)(x)dt.$$

- Since $\mathcal{E}(\widehat{a}) \geq 0$ and $\mathcal{E}(a) = 0$, then a minimizes $\mathcal{E}$. Is it unique?
- Given $d(x, y) = |x - y|$, introduce the family of measures

$$\varrho(t)(A) = (\mu(t) \otimes \mu(t))(d^{-1}(A)), \text{ for all } t \in [0, T] \text{ and}$$

$$\rho(A) = \frac{1}{T}\int_0^T \int_A s^2 d\varrho(t)(s)dt$$

$\varrho(t)(s)$ tells precisely how likely is that there are two indexes $i \neq j$ such that $|x_i(t) - x_j(t)| = s$, while $\rho(s)$ averages these likelihoods over the entire time frame $[0, T]$ (weighted by s^2).

- $\text{supp}(\rho) \subseteq [0, 2R]$.

The limit functional $\mathcal{E}$

- Natural candidate for the Γ-limit $\mathcal{E}$ of the $\mathcal{E}_N$: as μ is the uniform limit of the μ^N then we define

$$\mathcal{E}(\widehat{a}) = \frac{1}{T}\int_0^T \int_{\mathbb{R}^d} \left| \left(F[\widehat{a}] - F[a]\right) * \mu(t) \right|^2 d\mu(t)(x)dt.$$

- Since $\mathcal{E}(\widehat{a}) \geq 0$ and $\mathcal{E}(a) = 0$, then a minimizes $\mathcal{E}$. Is it unique?
- Given $d(x,y) = |x - y|$, introduce the family of measures

$$\varrho(t)(A) = (\mu(t) \otimes \mu(t))(d^{-1}(A)), \text{ for all } t \in [0,T] \text{ and}$$

$$\rho(A) = \frac{1}{T}\int_0^T \int_A s^2 d\varrho(t)(s)dt$$

  $\varrho(t)(s)$ tells precisely how likely is that there are two indexes $i \neq j$ such that $|x_i(t) - x_j(t)| = s$, while $\rho(s)$ averages these likelihoods over the entire time frame $[0,T]$ (weighted by s^2).
- $\mathrm{supp}(\rho) \subseteq [0, 2R]$.

The limit functional $\mathcal{E}$

- Natural candidate for the Γ-limit $\mathcal{E}$ of the $\mathcal{E}_N$: as μ is the uniform limit of the μ^N then we define

$$\mathcal{E}(\widehat{a}) = \frac{1}{T}\int_0^T \int_{\mathbb{R}^d} \left| \left(F[\widehat{a}] - F[a]\right) * \mu(t) \right|^2 d\mu(t)(x)dt.$$

- Since $\mathcal{E}(\widehat{a}) \geq 0$ and $\mathcal{E}(a) = 0$, then a minimizes $\mathcal{E}$. Is it unique?
- Given $d(x, y) = |x - y|$, introduce the family of measures

$$\varrho(t)(A) = (\mu(t) \otimes \mu(t))(d^{-1}(A)), \text{ for all } t \in [0, T] \text{ and}$$

$$\rho(A) = \frac{1}{T}\int_0^T \int_A s^2 d\varrho(t)(s)dt$$

  $\varrho(t)(s)$ tells precisely how likely is that there are two indexes $i \neq j$ such that $|x_i(t) - x_j(t)| = s$, while $\rho(s)$ averages these likelihoods over the entire time frame $[0, T]$ (weighted by s^2).

- $\text{supp}(\rho) \subseteq [0, 2R]$.

The limit functional $\mathcal{E}$

- Natural candidate for the Γ-limit $\mathcal{E}$ of the $\mathcal{E}_N$: as μ is the uniform limit of the μ^N then we define

$$\mathcal{E}(\widehat{a}) = \frac{1}{T}\int_0^T \int_{\mathbb{R}^d} \left| \left(F[\widehat{a}] - F[a]\right) * \mu(t) \right|^2 d\mu(t)(x)dt.$$

- Since $\mathcal{E}(\widehat{a}) \geq 0$ and $\mathcal{E}(a) = 0$, then a minimizes $\mathcal{E}$. Is it unique?
- Given $d(x, y) = |x - y|$, introduce the family of measures

$$\varrho(t)(A) = (\mu(t) \otimes \mu(t))(d^{-1}(A)), \text{ for all } t \in [0, T] \text{ and}$$

$$\rho(A) = \frac{1}{T}\int_0^T \int_A s^2 d\varrho(t)(s)dt$$

  $\varrho(t)(s)$ tells precisely how likely is that there are two indexes $i \neq j$ such that $|x_i(t) - x_j(t)| = s$, while $\rho(s)$ averages these likelihoods over the entire time frame $[0, T]$ (weighted by s^2).
- $\text{supp}(\rho) \subseteq [0, 2R]$.

The coercivity condition

- By Jensen or Hölder inequality it holds

$$\mathcal{E}(\widehat{a}) \leq \int_{\mathbb{R}_+} |\widehat{a}(s) - a(s)|^2 d\rho(s) = \|\widehat{a} - a\|^2_{L_2(\mathbb{R}_+,\rho)}$$

- If ADDITIONALLY $\mathcal{E}$ satisfies the coercivity condition

$$(1) \qquad \exists c_T > 0 \text{ such that } c_T \|\widehat{a} - a\|^2_{L_2(\mathbb{R}_+,\rho)} \leq \mathcal{E}(\widehat{a})$$

  then follows $\mathcal{E}(\widehat{a}) = 0 \implies \widehat{a} = a$ in $L_2(\mathbb{R}_+, \rho)$.
  Thus a is essentially the unique minimizer of $\mathcal{E}$ in $L_2(\mathbb{R}_+, \rho)$.
- The quantity $\rho(r)$ captures the frequency of the mutual distance r realized by two particles during the dynamics. If $\rho(r) = 0$, one cannot expect the reconstruction $\widehat{a}$ to agree with a at r.
- For several a and μ_0 one can verify (1) deterministically or with high probability. Numerical simulations confirm that $c_T > 0$ in many circumstances.

The coercivity condition

- By Jensen or Hölder inequality it holds

$$\mathcal{E}(\widehat{a}) \leq \int_{\mathbb{R}_+} |\widehat{a}(s) - a(s)|^2 d\rho(s) = \|\widehat{a} - a\|^2_{L_2(\mathbb{R}_+,\rho)}$$

- If ADDITIONALLY $\mathcal{E}$ satisfies the coercivity condition

(1) $\exists c_T > 0$ such that $c_T \|\widehat{a} - a\|^2_{L_2(\mathbb{R}_+,\rho)} \leq \mathcal{E}(\widehat{a})$

then follows $\mathcal{E}(\widehat{a}) = 0 \implies \widehat{a} = a$ in $L_2(\mathbb{R}_+, \rho)$.
Thus a is essentially the unique minimizer of $\mathcal{E}$ in $L_2(\mathbb{R}_+, \rho)$.

- The quantity $\rho(r)$ captures the frequency of the mutual distance r realized by two particles during the dynamics. If $\rho(r) = 0$, one cannot expect the reconstruction $\widehat{a}$ to agree with a at r.
- For several a and μ_0 one can verify (1) deterministically or with high probability. Numerical simulations confirm that $c_T > 0$ in many circumstances.

The coercivity condition

- By Jensen or Hölder inequality it holds

$$\mathcal{E}(\widehat{a}) \leq \int_{\mathbb{R}_+} |\widehat{a}(s) - a(s)|^2 d\rho(s) = \|\widehat{a} - a\|^2_{L_2(\mathbb{R}_+,\rho)}$$

- If ADDITIONALLY $\mathcal{E}$ satisfies the coercivity condition

(1) $\exists c_T > 0$ such that $c_T \|\widehat{a} - a\|^2_{L_2(\mathbb{R}_+,\rho)} \leq \mathcal{E}(\widehat{a})$

then follows $\mathcal{E}(\widehat{a}) = 0 \quad \Longrightarrow \quad \widehat{a} = a$ in $L_2(\mathbb{R}_+, \rho)$.
Thus a is essentially the unique minimizer of $\mathcal{E}$ in $L_2(\mathbb{R}_+, \rho)$.

- The quantity $\rho(r)$ captures the frequency of the mutual distance r realized by two particles during the dynamics. If $\rho(r) = 0$, one cannot expect the reconstruction $\widehat{a}$ to agree with a at r.
- For several a and μ_0 one can verify (1) deterministically or with high probability. Numerical simulations confirm that $c_T > 0$ in many circumstances.

The coercivity condition

- By Jensen or Hölder inequality it holds

$$\mathcal{E}(\widehat{a}) \leq \int_{\mathbb{R}_+} |\widehat{a}(s) - a(s)|^2 d\rho(s) = \|\widehat{a} - a\|^2_{L_2(\mathbb{R}_+,\rho)}$$

- If ADDITIONALLY $\mathcal{E}$ satisfies the coercivity condition

  (1) $\exists c_T > 0$ such that $c_T \|\widehat{a} - a\|^2_{L_2(\mathbb{R}_+,\rho)} \leq \mathcal{E}(\widehat{a})$

  then follows $\mathcal{E}(\widehat{a}) = 0 \implies \widehat{a} = a$ in $L_2(\mathbb{R}_+, \rho)$.
  Thus a is essentially the unique minimizer of $\mathcal{E}$ in $L_2(\mathbb{R}_+, \rho)$.
- The quantity $\rho(r)$ captures the frequency of the mutual distance r realized by two particles during the dynamics. If $\rho(r) = 0$, one cannot expect the reconstruction $\widehat{a}$ to agree with a at r.
- For several a and μ_0 one can verify (1) deterministically or with high probability. Numerical simulations confirm that $c_T > 0$ in many circumstances.

The coercivity condition

- By Jensen or Hölder inequality it holds

$$\mathcal{E}(\widehat{a}) \leq \int_{\mathbb{R}_+} |\widehat{a}(s) - a(s)|^2 d\rho(s) = \|\widehat{a} - a\|^2_{L_2(\mathbb{R}_+,\rho)}$$

- If ADDITIONALLY $\mathcal{E}$ satisfies the coercivity condition

$$(1) \qquad \exists c_T > 0 \text{ such that } c_T \|\widehat{a} - a\|^2_{L_2(\mathbb{R}_+,\rho)} \leq \mathcal{E}(\widehat{a})$$

  then follows $\mathcal{E}(\widehat{a}) = 0 \implies \widehat{a} = a$ in $L_2(\mathbb{R}_+, \rho)$.
  Thus a is essentially the unique minimizer of $\mathcal{E}$ in $L_2(\mathbb{R}_+, \rho)$.
- The quantity $\rho(r)$ captures the frequency of the mutual distance r realized by two particles during the dynamics. If $\rho(r) = 0$, one cannot expect the reconstruction $\widehat{a}$ to agree with a at r.
- For several a and μ_0 one can verify (1) deterministically or with high probability. Numerical simulations confirm that $c_T > 0$ in many circumstances.

Existence of minimizers of $\mathcal{E}_N$

Proposition Fix $M > 0$ and $K = [0, 2R] \subset \mathbb{R}_+$ for some $R > 0$. Then the set

$$X_{M,K} = \{b \in W^1_\infty(K) : \|b\|_{L_\infty(K)} + \|b'\|_{L_\infty(K)} \leq M\}$$

is relatively compact with respect to the uniform convergence on K.

Proposition Assume $a \in X$. Let V be a closed subset of $X_{M,K}$ w.r.t. the uniform convergence. Then

$$\operatorname*{argmin}_{\widehat{a} \in V} \mathcal{E}_N(\widehat{a}) \neq \emptyset.$$

Definition The closed subsets $V_N \subset X_{M,K}$, $N \in \mathbb{N}$ have the uniform approximation property in $L_\infty(K)$ if for all $b \in X_{M,K}$ there exists $(b_N)_{N \in \mathbb{N}}$ such that

- $b_N \in V_N$ for every $N \in \mathbb{N}$ and
- $(b_N)_{N \in \mathbb{N}}$ converges uniformly to b on K.

Existence of minimizers of $\mathcal{E}_N$

Proposition Fix $M > 0$ and $K = [0, 2R] \subset \mathbb{R}_+$ for some $R > 0$. Then the set

$$X_{M,K} = \{b \in W^1_\infty(K) : \|b\|_{L_\infty(K)} + \|b'\|_{L_\infty(K)} \leq M\}$$

is relatively compact with respect to the uniform convergence on K.

Proposition Assume $a \in X$. Let V be a closed subset of $X_{M,K}$ w.r.t. the uniform convergence. Then

$$\operatorname*{argmin}_{\widehat{a} \in V} \mathcal{E}_N(\widehat{a}) \neq \emptyset.$$

Definition The closed subsets $V_N \subset X_{M,K}$, $N \in \mathbb{N}$ have the uniform approximation property in $L_\infty(K)$ if for all $b \in X_{M,K}$ there exists $(b_N)_{N \in \mathbb{N}}$ such that

- $b_N \in V_N$ for every $N \in \mathbb{N}$ and
- $(b_N)_{N \in \mathbb{N}}$ converges uniformly to b on K.

Existence of minimizers of $\mathcal{E}_N$

Proposition Fix $M > 0$ and $K = [0, 2R] \subset \mathbb{R}_+$ for some $R > 0$. Then the set

$$X_{M,K} = \{b \in W^1_\infty(K) : \|b\|_{L_\infty(K)} + \|b'\|_{L_\infty(K)} \leq M\}$$

is relatively compact with respect to the uniform convergence on K.

Proposition Assume $a \in X$. Let V be a closed subset of $X_{M,K}$ w.r.t. the uniform convergence. Then

$$\operatorname*{argmin}_{\widehat{a} \in V} \mathcal{E}_N(\widehat{a}) \neq \emptyset.$$

Definition The closed subsets $V_N \subset X_{M,K}$, $N \in \mathbb{N}$ have the uniform approximation property in $L_\infty(K)$ if for all $b \in X_{M,K}$ there exists $(b_N)_{N\in\mathbb{N}}$ such that

- $b_N \in V_N$ for every $N \in \mathbb{N}$ and
- $(b_N)_{N\in\mathbb{N}}$ converges uniformly to b on K.

Existence of minimizers of $\mathcal{E}_N$

Proposition Fix $M > 0$ and $K = [0, 2R] \subset \mathbb{R}_+$ for some $R > 0$. Then the set

$$X_{M,K} = \{b \in W^1_\infty(K) : \|b\|_{L_\infty(K)} + \|b'\|_{L_\infty(K)} \leq M\}$$

is relatively compact with respect to the uniform convergence on K.

Proposition Assume $a \in X$. Let V be a closed subset of $X_{M,K}$ w.r.t. the uniform convergence. Then

$$\operatorname*{argmin}_{\widehat{a} \in V} \mathcal{E}_N(\widehat{a}) \neq \emptyset.$$

Definition The closed subsets $V_N \subset X_{M,K}$, $N \in \mathbb{N}$ have the uniform approximation property in $L_\infty(K)$ if for all $b \in X_{M,K}$ there exists $(b_N)_{N \in \mathbb{N}}$ such that

- $b_N \in V_N$ for every $N \in \mathbb{N}$ and
- $(b_N)_{N \in \mathbb{N}}$ converges uniformly to b on K.

Existence of minimizers of $\mathcal{E}_N$

Proposition Fix $M > 0$ and $K = [0, 2R] \subset \mathbb{R}_+$ for some $R > 0$. Then the set

$$X_{M,K} = \{b \in W^1_\infty(K) : \|b\|_{L_\infty(K)} + \|b'\|_{L_\infty(K)} \leq M\}$$

is relatively compact with respect to the uniform convergence on K.

Proposition Assume $a \in X$. Let V be a closed subset of $X_{M,K}$ w.r.t. the uniform convergence. Then

$$\operatorname*{argmin}_{\widehat{a} \in V} \mathcal{E}_N(\widehat{a}) \neq \emptyset.$$

Definition The closed subsets $V_N \subset X_{M,K}$, $N \in \mathbb{N}$ have the uniform approximation property in $L_\infty(K)$ if for all $b \in X_{M,K}$ there exists $(b_N)_{N \in \mathbb{N}}$ such that

- $b_N \in V_N$ for every $N \in \mathbb{N}$ and
- $(b_N)_{N \in \mathbb{N}}$ converges uniformly to b on K.

The Γ-convergence - I

Theorem Assume $a \in X$, fix $\mu_0 \in \mathcal{P}_c(\mathbb{R}^d)$ and set

$$M \geq \|a\|_{L_\infty(K)} + \|a'\|_{L_\infty(K)}.$$

For every $N \in \mathbb{N}$, let $x^N_{0,1}, \ldots, x^N_{0,N}$ be i.i. μ_0-distributed and define

$$\mathcal{E}_N(\widehat{a}) = \frac{1}{T}\int_0^T \int_{\mathbb{R}^d} \left| \left(F[\widehat{a}] - F[a]\right) * \mu^N(t) \right|^2 d\mu^N(t)(x)dt,$$

where μ^N is the solution of c.e. with initial datum $\mu^N_0 = \frac{1}{N}\sum_{i=1}^N \delta_{x^N_{0,i}}$.
Let $V_N \subset X_{M,K}$ be a sequence with the uniform approximation property and consider

$$\widehat{a}_N \in \operatorname*{argmin}_{\widehat{a} \in V_N} \mathcal{E}_N(\widehat{a}).$$

Then $(\widehat{a}_N)_{N\in\mathbb{N}}$ converges uniformly on K (up to subsequences) to some continuous function $\widehat{a} \in X_{M,K}$ such that $\mathcal{E}(\widehat{a}) = 0$.
Furthermore...

The Γ-convergence - I

Theorem Assume $a \in X$, fix $\mu_0 \in \mathcal{P}_c(\mathbb{R}^d)$ and set

$$M \geq \|a\|_{L_\infty(K)} + \|a'\|_{L_\infty(K)}.$$

For every $N \in \mathbb{N}$, let $x^N_{0,1}, \dots, x^N_{0,N}$ be i.i. μ_0-distributed and define

$$\mathcal{E}_N(\widehat{a}) = \frac{1}{T}\int_0^T \int_{\mathbb{R}^d} \left| \left(F[\widehat{a}] - F[a]\right) * \mu^N(t) \right|^2 d\mu^N(t)(x)dt,$$

where μ^N is the solution of c.e. with initial datum $\mu^N_0 = \frac{1}{N}\sum_{i=1}^N \delta_{x^N_{0,i}}$.

Let $V_N \subset X_{M,K}$ be a sequence with the uniform approximation property and consider

$$\widehat{a}_N \in \operatorname*{argmin}_{\widehat{a} \in V_N} \mathcal{E}_N(\widehat{a}).$$

Then $(\widehat{a}_N)_{N\in\mathbb{N}}$ converges uniformly on K (up to subsequences) to some continuous function $\widehat{a} \in X_{M,K}$ such that $\mathcal{E}(\widehat{a}) = 0$.

Furthermore...

The Γ-convergence - I

Theorem Assume $a \in X$, fix $\mu_0 \in \mathcal{P}_c(\mathbb{R}^d)$ and set

$$M \geq \|a\|_{L_\infty(K)} + \|a'\|_{L_\infty(K)}.$$

For every $N \in \mathbb{N}$, let $x^N_{0,1}, \dots, x^N_{0,N}$ be i.i. μ_0-distributed and define

$$\mathcal{E}_N(\widehat{a}) = \frac{1}{T}\int_0^T \int_{\mathbb{R}^d} \left| \left(F[\widehat{a}] - F[a]\right) * \mu^N(t)\right|^2 d\mu^N(t)(x)dt,$$

where μ^N is the solution of c.e. with initial datum $\mu_0^N = \frac{1}{N}\sum_{i=1}^N \delta_{x^N_{0,i}}$.
Let $V_N \subset X_{M,K}$ be a sequence with the uniform approximation property and consider

$$\widehat{a}_N \in \underset{\widehat{a} \in V_N}{\operatorname{argmin}}\, \mathcal{E}_N(\widehat{a}).$$

Then $(\widehat{a}_N)_{N\in\mathbb{N}}$ converges uniformly on K (up to subsequences) to some continuous function $\widehat{a} \in X_{M,K}$ such that $\mathcal{E}(\widehat{a}) = 0$.
Furthermore...

The Γ-convergence - I

Theorem Assume $a \in X$, fix $\mu_0 \in \mathcal{P}_c(\mathbb{R}^d)$ and set

$$M \geq \|a\|_{L_\infty(K)} + \|a'\|_{L_\infty(K)}.$$

For every $N \in \mathbb{N}$, let $x_{0,1}^N, \dots, x_{0,N}^N$ be i.i. μ_0-distributed and define

$$\mathcal{E}_N(\widehat{a}) = \frac{1}{T}\int_0^T \int_{\mathbb{R}^d} \left|\left(F[\widehat{a}] - F[a]\right) * \mu^N(t)\right|^2 d\mu^N(t)(x)dt,$$

where μ^N is the solution of c.e. with initial datum $\mu_0^N = \frac{1}{N}\sum_{i=1}^N \delta_{x_{0,i}^N}$.

Let $V_N \subset X_{M,K}$ be a sequence with the uniform approximation property and consider

$$\widehat{a}_N \in \operatorname*{argmin}_{\widehat{a} \in V_N} \mathcal{E}_N(\widehat{a}).$$

Then $(\widehat{a}_N)_{N \in \mathbb{N}}$ converges uniformly on K (up to subsequences) to some continuous function $\widehat{a} \in X_{M,K}$ such that $\mathcal{E}(\widehat{a}) = 0$.

Furthermore...

The Γ-convergence - I

Theorem Assume $a \in X$, fix $\mu_0 \in \mathcal{P}_c(\mathbb{R}^d)$ and set

$$M \geq \|a\|_{L_\infty(K)} + \|a'\|_{L_\infty(K)}.$$

For every $N \in \mathbb{N}$, let $x^N_{0,1}, \ldots, x^N_{0,N}$ be i.i. μ_0-distributed and define

$$\mathcal{E}_N(\widehat{a}) = \frac{1}{T}\int_0^T \int_{\mathbb{R}^d} \left| \left(F[\widehat{a}] - F[a]\right) * \mu^N(t) \right|^2 d\mu^N(t)(x)dt,$$

where μ^N is the solution of c.e. with initial datum $\mu^N_0 = \frac{1}{N}\sum_{i=1}^N \delta_{x^N_{0,i}}$.
Let $V_N \subset X_{M,K}$ be a sequence with the uniform approximation property and consider

$$\widehat{a}_N \in \operatorname*{argmin}_{\widehat{a} \in V_N} \mathcal{E}_N(\widehat{a}).$$

Then $(\widehat{a}_N)_{N\in\mathbb{N}}$ converges uniformly on K (up to subsequences) to some continuous function $\widehat{a} \in X_{M,K}$ such that $\mathcal{E}(\widehat{a}) = 0$.
Furthermore...

The Γ-convergence - II

...if the coercivity condition holds, then $\widehat{a} = a$ in $L_2(\mathbb{R}_+, \rho)$ and

$$\|\widehat{a}_N - a\|_{L_2(\mathbb{R}_+,\rho)} \leq C(M, T, \mu_0) N^{-1}.$$

Proof: by compactness of $X_{M,K}$, the sequence of minimizers $(\widehat{a}_N)_{N\in\mathbb{N}}$ admits a subsequence converging to some $\widehat{a} \in X_{M,K}$. The uniform approximation property of the V_N implies $\mathcal{E}(b) \geq \mathcal{E}(\widehat{a})$ for all $b \in X_{M,K}$, whence $0 = \mathcal{E}(a) \geq \mathcal{E}(\widehat{a}) \geq 0$. □

Problem: the bound M depends on a, but a is unknown!

The Γ-convergence - II

...if the coercivity condition holds, then $\widehat{a} = a$ in $L_2(\mathbb{R}_+, \rho)$ and

$$\|\widehat{a}_N - a\|_{L_2(\mathbb{R}_+,\rho)} \leq C(M, T, \mu_0) N^{-1}.$$

Proof: by compactness of $X_{M,K}$, the sequence of minimizers $(\widehat{a}_N)_{N\in\mathbb{N}}$ admits a subsequence converging to some $\widehat{a} \in X_{M,K}$. The uniform approximation property of the V_N implies $\mathcal{E}(b) \geq \mathcal{E}(\widehat{a})$ for all $b \in X_{M,K}$, whence $0 = \mathcal{E}(a) \geq \mathcal{E}(\widehat{a}) \geq 0$. □

Problem: the bound M depends on a, but a is unknown!

The Γ-convergence - II

...if the coercivity condition holds, then $\widehat{a} = a$ in $L_2(\mathbb{R}_+, \rho)$ and

$$\|\widehat{a}_N - a\|_{L_2(\mathbb{R}_+,\rho)} \leq C(M, T, \mu_0)N^{-1}.$$

Proof: by compactness of $X_{M,K}$, the sequence of minimizers $(\widehat{a}_N)_{N\in\mathbb{N}}$ admits a subsequence converging to some $\widehat{a} \in X_{M,K}$. The uniform approximation property of the V_N implies $\mathcal{E}(b) \geq \mathcal{E}(\widehat{a})$ for all $b \in X_{M,K}$, whence $0 = \mathcal{E}(a) \geq \mathcal{E}(\widehat{a}) \geq 0$. □

Problem: the bound M depends on a, but a is unknown!

The Γ-convergence - II

...if the coercivity condition holds, then $\widehat{a} = a$ in $L_2(\mathbb{R}_+, \rho)$ and

$$\|\widehat{a}_N - a\|_{L_2(\mathbb{R}_+,\rho)} \leq C(M, T, \mu_0) N^{-1}.$$

Proof: by compactness of $X_{M,K}$, the sequence of minimizers $(\widehat{a}_N)_{N\in\mathbb{N}}$ admits a subsequence converging to some $\widehat{a} \in X_{M,K}$. The uniform approximation property of the V_N implies $\mathcal{E}(b) \geq \mathcal{E}(\widehat{a})$ for all $b \in X_{M,K}$, whence $0 = \mathcal{E}(a) \geq \mathcal{E}(\widehat{a}) \geq 0$. □

Problem: the bound M depends on a, but a is unknown!

The Γ-convergence - II

...if the coercivity condition holds, then $\widehat{a} = a$ in $L_2(\mathbb{R}_+, \rho)$ and

$$\|\widehat{a}_N - a\|_{L_2(\mathbb{R}_+,\rho)} \leq C(M, T, \mu_0)N^{-1}.$$

Proof: by compactness of $X_{M,K}$, the sequence of minimizers $(\widehat{a}_N)_{N\in\mathbb{N}}$ admits a subsequence converging to some $\widehat{a} \in X_{M,K}$. The uniform approximation property of the V_N implies $\mathcal{E}(b) \geq \mathcal{E}(\widehat{a})$ for all $b \in X_{M,K}$, whence $0 = \mathcal{E}(a) \geq \mathcal{E}(\widehat{a}) \geq 0$. □

Problem: the bound M depends on a, but a is unknown!

The Γ-convergence - II

...if the coercivity condition holds, then $\widehat{a} = a$ in $L_2(\mathbb{R}_+, \rho)$ and

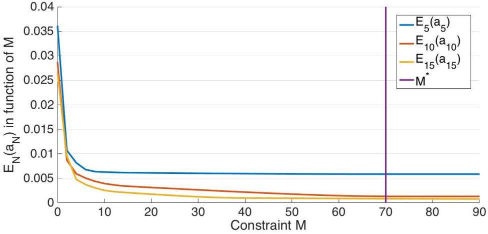
$$\|\widehat{a}_N - a\|_{L_2(\mathbb{R}_+,\rho)} \leq C(M, T, \mu_0) N^{-1}.$$

Proof: by compactness of $X_{M,K}$, the sequence of minimizers $(\widehat{a}_N)_{N\in\mathbb{N}}$ admits a subsequence converging to some $\widehat{a} \in X_{M,K}$. The uniform approximation property of the V_N implies $\mathcal{E}(b) \geq \mathcal{E}(\widehat{a})$ for all $b \in X_{M,K}$, whence $0 = \mathcal{E}(a) \geq \mathcal{E}(\widehat{a}) \geq 0$. □

Problem: the bound M depends on a, but a is unknown!

Solution: For N fixed, $\mathcal{E}_N(\widehat{a}_N)$ is decreasing for $M \to +\infty$ and $\exists M^*$ independent from N such that

$$\text{“}\frac{\partial \mathcal{E}_N(\widehat{a}_N)}{\partial M}(M^*) = 0\text{”}.$$



Minimizing $\mathcal{E}_N$ is a least squares minimization

The advantage of minimizing $\mathcal{E}_N(\widehat{a})$ is that it can be reduced to a simple ℓ_2 minimization. Indeed

- let $V_N = \mathrm{span}\{\varphi_\lambda\}_{\lambda=1}^{D(N)}$ where the φ_λ are a linear B-spline basis with $D(N)$ elements supported on $[0, 2R]$,
- let $0 = t_0 < t_1 < \ldots < t_m = T$ be a time discretization,
- let $\dot{x}_i(t_k) = \frac{x_i(t_k) - x_i(t_{k-1})}{t_k - t_{k-1}}$, for every $k \geq 1$ be the finite differences approximating the true velocities,

then the *discrete-time error functional* satisfies

$$\overline{\mathcal{E}}_N(\widehat{a}) = \frac{1}{m}\sum_{k=1}^{m}\frac{1}{N}\sum_{j=1}^{N}\left|\sum_{\lambda=1}^{D(N)}\frac{a_\lambda}{N}\sum_{i=1}^{N}\varphi_\lambda(|x_j(t_k) - x_i(t_k)|)(x_j(t_k) - x_i(t_k)) - \dot{x}_i(t_k)\right|^2$$

$$= \frac{1}{mN}\|C\vec{a} - v\|_2^2.$$

where $\vec{a} = (a_1, \ldots, a_{D(N)})$, $v = (\dot{x}_1(t_1), \ldots, \dot{x}_N(t_1), \ldots, \dot{x}_1(t_m), \ldots, \dot{x}_N(t_m))$.

Minimizing $\mathcal{E}_N$ is a least squares minimization

The advantage of minimizing $\mathcal{E}_N(\widehat{a})$ is that it can be reduced to a simple ℓ_2 minimization. Indeed

- let $V_N = \mathrm{span}\{\varphi_\lambda\}_{\lambda=1}^{D(N)}$ where the φ_λ are a linear B-spline basis with $D(N)$ elements supported on $[0, 2R]$,
- let $0 = t_0 < t_1 < \ldots < t_m = T$ be a time discretization,
- let $\dot{x}_i(t_k) = \frac{x_i(t_k) - x_i(t_{k-1})}{t_k - t_{k-1}}$, for every $k \geq 1$ be the finite differences approximating the true velocities,

then the *discrete-time error functional* satisfies

$$\overline{\mathcal{E}}_N(\widehat{a}) = \frac{1}{m}\sum_{k=1}^{m}\frac{1}{N}\sum_{j=1}^{N}\left|\sum_{\lambda=1}^{D(N)}\frac{a_\lambda}{N}\sum_{i=1}^{N}\varphi_\lambda(|x_j(t_k)-x_i(t_k)|)(x_j(t_k)-x_i(t_k)) - \dot{x}_i(t_k)\right|^2$$

$$= \frac{1}{mN}\|C\vec{a} - v\|_2^2.$$

where $\vec{a} = (a_1, \ldots, a_{D(N)})$, $v = (\dot{x}_1(t_1), \ldots, \dot{x}_N(t_1), \ldots, \dot{x}_1(t_m), \ldots, \dot{x}_N(t_m))$.

Minimizing $\mathcal{E}_N$ is a least squares minimization

The advantage of minimizing $\mathcal{E}_N(\widehat{a})$ is that it can be reduced to a simple ℓ_2 minimization. Indeed

- let $V_N = \text{span}\{\varphi_\lambda\}_{\lambda=1}^{D(N)}$ where the φ_λ are a linear B-spline basis with $D(N)$ elements supported on $[0, 2R]$,
- let $0 = t_0 < t_1 < \ldots < t_m = T$ be a time discretization,
- let $\dot{x}_i(t_k) = \frac{x_i(t_k) - x_i(t_{k-1})}{t_k - t_{k-1}}$, for every $k \geq 1$ be the finite differences approximating the true velocities,

then the *discrete-time error functional* satisfies

$$\overline{\mathcal{E}}_N(\widehat{a}) = \frac{1}{m}\sum_{k=1}^{m}\frac{1}{N}\sum_{j=1}^{N}\left|\sum_{\lambda=1}^{D(N)}\frac{a_\lambda}{N}\sum_{i=1}^{N}\varphi_\lambda(|x_j(t_k) - x_i(t_k)|)(x_j(t_k) - x_i(t_k)) - \dot{x}_i(t_k)\right|^2$$

$$= \frac{1}{mN}\|C\vec{a} - v\|_2^2.$$

where $\vec{a} = (a_1, \ldots, a_{D(N)})$, $v = (\dot{x}_1(t_1), \ldots, \dot{x}_N(t_1), \ldots, \dot{x}_1(t_m), \ldots, \dot{x}_N(t_m))$.

Minimizing $\mathcal{E}_N$ is a least squares minimization

The advantage of minimizing $\mathcal{E}_N(\widehat{a})$ is that it can be reduced to a simple ℓ_2 minimization. Indeed

- let $V_N = \mathrm{span}\{\varphi_\lambda\}_{\lambda=1}^{D(N)}$ where the φ_λ are a linear B-spline basis with $D(N)$ elements supported on $[0, 2R]$,
- let $0 = t_0 < t_1 < \ldots < t_m = T$ be a time discretization,
- let $\dot{x}_i(t_k) = \frac{x_i(t_k) - x_i(t_{k-1})}{t_k - t_{k-1}}$, for every $k \geq 1$ be the finite differences approximating the true velocities,

then the *discrete-time error functional* satisfies

$$\overline{\mathcal{E}}_N(\widehat{a}) = \frac{1}{m}\sum_{k=1}^{m} \frac{1}{N}\sum_{j=1}^{N} \left| \sum_{\lambda=1}^{D(N)} \frac{a_\lambda}{N} \sum_{i=1}^{N} \varphi_\lambda(|x_j(t_k) - x_i(t_k)|)(x_j(t_k) - x_i(t_k)) - \dot{x}_i(t_k) \right|^2$$

$$= \frac{1}{mN}\|C\vec{a} - v\|_2^2.$$

where $\vec{a} = (a_1, \ldots, a_{D(N)})$, $v = (\dot{x}_1(t_1), \ldots, \dot{x}_N(t_1), \ldots, \dot{x}_1(t_m), \ldots, \dot{x}_N(t_m))$.

Minimizing $\mathcal{E}_N$ is a least squares minimization

The advantage of minimizing $\mathcal{E}_N(\widehat{a})$ is that it can be reduced to a simple ℓ_2 minimization. Indeed

- let $V_N = \mathrm{span}\{\varphi_\lambda\}_{\lambda=1}^{D(N)}$ where the φ_λ are a linear B-spline basis with $D(N)$ elements supported on $[0, 2R]$,
- let $0 = t_0 < t_1 < \ldots < t_m = T$ be a time discretization,
- let $\dot{x}_i(t_k) = \frac{x_i(t_k) - x_i(t_{k-1})}{t_k - t_{k-1}}$, for every $k \geq 1$ be the finite differences approximating the true velocities,

then the *discrete-time error functional* satisfies

$$\overline{\mathcal{E}}_N(\widehat{a}) = \frac{1}{m}\sum_{k=1}^{m}\frac{1}{N}\sum_{j=1}^{N}\left|\sum_{\lambda=1}^{D(N)}\frac{a_\lambda}{N}\sum_{i=1}^{N}\varphi_\lambda(|x_j(t_k) - x_i(t_k)|)(x_j(t_k) - x_i(t_k)) - \dot{x}_i(t_k)\right|^2$$

$$= \frac{1}{mN}\|C\vec{a} - v\|_2^2.$$

where $\vec{a} = (a_1, \ldots, a_{D(N)})$, $v = (\dot{x}_1(t_1), \ldots, \dot{x}_N(t_1), \ldots, \dot{x}_1(t_m), \ldots, \dot{x}_N(t_m))$.

Minimizing $\mathcal{E}_N$ is a least squares minimization

The advantage of minimizing $\mathcal{E}_N(\widehat{a})$ is that it can be reduced to a simple ℓ_2 minimization. Indeed

- let $V_N = \text{span}\{\varphi_\lambda\}_{\lambda=1}^{D(N)}$ where the φ_λ are a linear B-spline basis with $D(N)$ elements supported on $[0, 2R]$,
- let $0 = t_0 < t_1 < \ldots < t_m = T$ be a time discretization,
- let $\dot{x}_i(t_k) = \frac{x_i(t_k) - x_i(t_{k-1})}{t_k - t_{k-1}}$, for every $k \geq 1$ be the finite differences approximating the true velocities,

then the *discrete-time error functional* satisfies

$$\overline{\mathcal{E}}_N(\widehat{a}) = \frac{1}{m}\sum_{k=1}^{m}\frac{1}{N}\sum_{j=1}^{N}\left|\sum_{\lambda=1}^{D(N)}\frac{a_\lambda}{N}\sum_{i=1}^{N}\varphi_\lambda(|x_j(t_k) - x_i(t_k)|)(x_j(t_k) - x_i(t_k)) - \dot{x}_i(t_k)\right|^2$$

$$= \frac{1}{mN}\|C\vec{a} - v\|_2^2.$$

where $\vec{a} = (a_1, \ldots, a_{D(N)})$, $v = (\dot{x}_1(t_1), \ldots, \dot{x}_N(t_1), \ldots, \dot{x}_1(t_m), \ldots, \dot{x}_N(t_m))$.

Minimizing $\mathcal{E}_N$ is a least squares minimization

The advantage of minimizing $\mathcal{E}_N(\widehat{a})$ is that it can be reduced to a simple ℓ_2 minimization. Indeed

- let $V_N = \mathrm{span}\{\varphi_\lambda\}_{\lambda=1}^{D(N)}$ where the φ_λ are a linear B-spline basis with $D(N)$ elements supported on $[0, 2R]$,
- let $0 = t_0 < t_1 < \ldots < t_m = T$ be a time discretization,
- let $\dot{x}_i(t_k) = \frac{x_i(t_k) - x_i(t_{k-1})}{t_k - t_{k-1}}$, for every $k \geq 1$ be the finite differences approximating the true velocities,

then the *discrete-time error functional* satisfies

$$\overline{\mathcal{E}}_N(\widehat{a}) = \frac{1}{m}\sum_{k=1}^{m}\frac{1}{N}\sum_{j=1}^{N}\left|\sum_{\lambda=1}^{D(N)}\frac{a_\lambda}{N}\sum_{i=1}^{N}\varphi_\lambda(|x_j(t_k) - x_i(t_k)|)(x_j(t_k) - x_i(t_k)) - \dot{x}_i(t_k)\right|^2$$
$$= \frac{1}{mN}\|C\vec{a} - v\|_2^2.$$

where $\vec{a} = (a_1, \ldots, a_{D(N)})$, $v = (\dot{x}_1(t_1), \ldots, \dot{x}_N(t_1), \ldots, \dot{x}_1(t_m), \ldots, \dot{x}_N(t_m))$.

How we implement the constraints

If $D = \begin{bmatrix} 1 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ & & & & \\ 0 & 0 & \dots & 1 & -1 \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix}$ then

$$\|a\|_{L_\infty([0,2R])} \leq 2\|\vec{a}\|_\infty \text{ and } \|a'\|_{L_\infty([0,2R])} \leq \|D\vec{a}\|_\infty,$$

hence we numerically implement the convex constrained minimization

$$\min_{\widehat{a} \in V_N} \mathcal{E}_N(\widehat{a}) \quad \text{subject to} \quad \|\widehat{a}\|_{L_\infty([0,R])} + \|\widehat{a}'\|_{L_\infty([0,R])} \leq M,$$

in the following way

$$\min_{\vec{a} \in \mathbb{R}^{D(N)}} \frac{1}{mN} \|C\vec{a} - v\|_2^2 \quad \text{subject to} \quad 2\|\vec{a}\|_\infty + \|D\vec{a}\|_\infty \leq M.$$

How we implement the constraints

If $D = \begin{bmatrix} 1 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \\ 0 & 0 & \dots & 1 & -1 \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix}$ then

$$\|a\|_{L_\infty([0,2R])} \leq 2\|\vec{a}\|_\infty \text{ and } \|a'\|_{L_\infty([0,2R])} \leq \|D\vec{a}\|_\infty,$$

hence we numerically implement the convex constrained minimization

$$\min_{\widehat{a} \in V_N} \mathcal{E}_N(\widehat{a}) \quad \text{subject to} \quad \|\widehat{a}\|_{L_\infty([0,R])} + \|\widehat{a}'\|_{L_\infty([0,R])} \leq M,$$

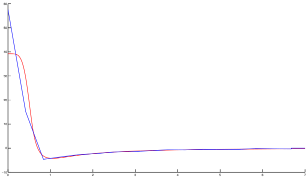
in the following way

$$\min_{\vec{a} \in \mathbb{R}^{D(N)}} \frac{1}{mN} \|C\vec{a} - v\|_2^2 \quad \text{subject to} \quad 2\|\vec{a}\|_\infty + \|D\vec{a}\|_\infty \leq M.$$

Varying N - I

d	L	T	M	N	$D(N)$
2	3	0.5	100	$[10, 20, 40, 80]$	$2N$

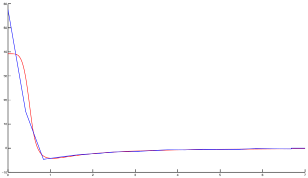
Table: Parameter values

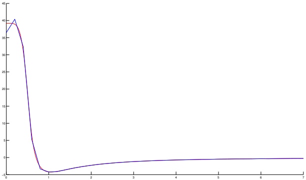


Varying N - I

d	L	T	M	N	$D(N)$
2	3	0.5	100	$[10, 20, 40, 80]$	$2N$

Table: Parameter values

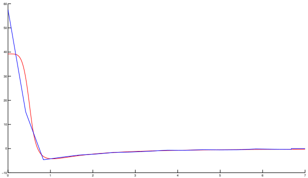


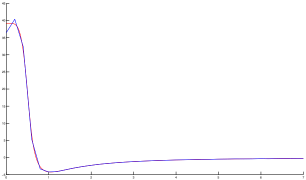


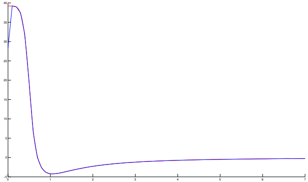
Varying N - I

d	L	T	M	N	$D(N)$
2	3	0.5	100	$[10, 20, 40, 80]$	$2N$

Table: Parameter values



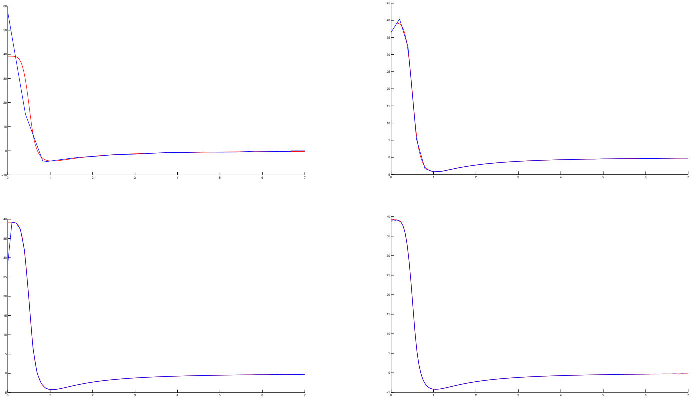




Varying N - I

d	L	T	M	N	$D(N)$
2	3	0.5	100	$[10, 20, 40, 80]$	$2N$

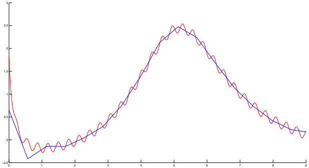
Table: Parameter values



Varying N - II

d	L	T	M	N	$D(N)$
2	3	0.5	100	$[10, 20, 40, 80]$	$2N$

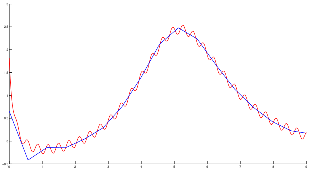
Table: Parameter values

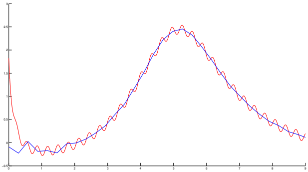


Varying N - II

d	L	T	M	N	$D(N)$
2	3	0.5	100	$[10, 20, 40, 80]$	$2N$

Table: Parameter values

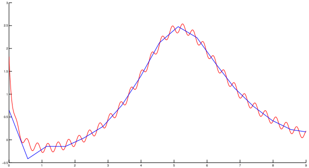


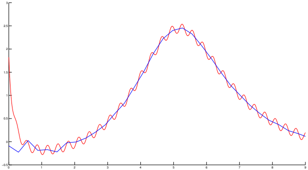


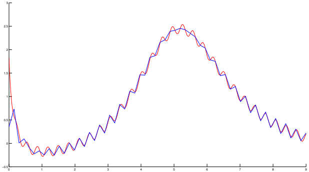
Varying N - II

d	L	T	M	N	$D(N)$
2	3	0.5	100	$[10, 20, 40, 80]$	$2N$

Table: Parameter values



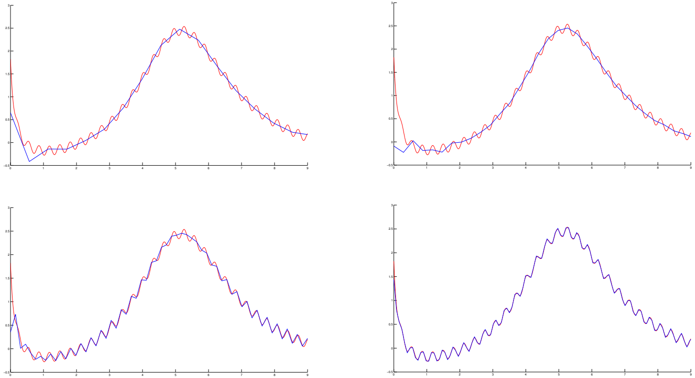




Varying N - II

d	L	T	M	N	$D(N)$
2	3	0.5	100	$[10, 20, 40, 80]$	$2N$

Table: Parameter values



The coercivity condition

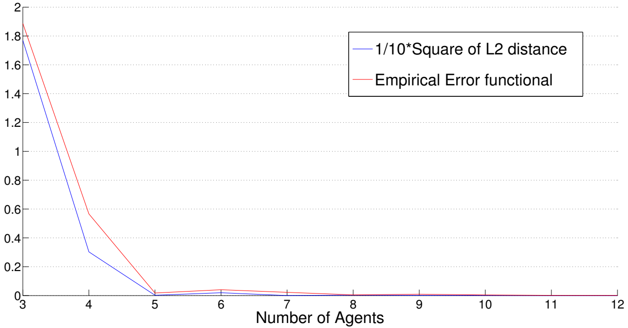
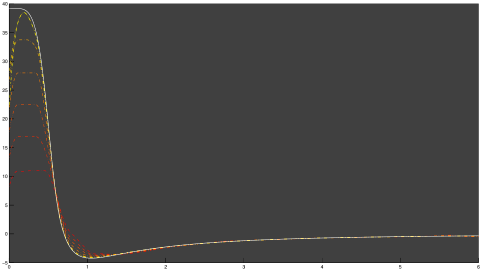


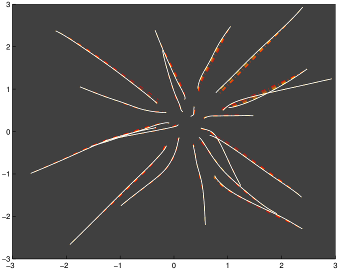
Figure: Plot of $\frac{1}{10}\|a - \widehat{a}_N\|^2_{L_2(\mathbb{R}_+,\rho)}$ and $\overline{\mathcal{E}}_N(\widehat{a}_N)$. We can estimate the constant c_T with the value $\frac{1}{10}$.

d	T	M	N	$D(N)$
2	0.5	100	$[3, 4, \dots, 12]$	$3N - 5$

Tuning the constraint M - I

Left: reconstruction of a kernel with different M.
Right: reconstruction of agents' trajectories with different M.
In white: true kernel and true trajectories.
The brighter the reconstruction, the bigger M.

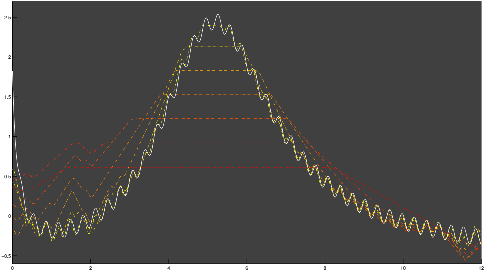


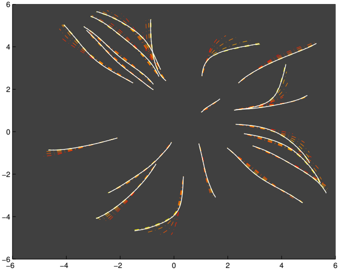


d	L	T	M	N	$D(N)$
2	3	1	$2.7 \times [10, 15, \ldots, 40]$	20	60

Tuning the constraint M - II

Left: reconstruction of a kernel with different M.
Right: reconstruction of agents' trajectories with different M.
In white: true kernel and true trajectories.
The brighter the reconstruction, the bigger M.





d	L	T	M	N	$D(N)$
2	3	1	$1.25 \times [10, 15, \ldots, 40]$	20	150

A few info

- **WWW:** http://www-m15.ma.tum.de/Allgemeines/MattiaBongini

- **References**:

  - M. Bongini, M. Fornasier, M. Hansen, and M. Maggioni. *Inferring interaction rules from observations of evolutive systems I: The variational approach.* in preparation, 2016.
  - M. Bongini, M. Fornasier, M. Hansen, and M. Maggioni. *Inferring interaction rules from observations of evolutive systems II: The universal learning approach.* in preparation, 2016.
  - G. Albi, M. Bongini, E. Cristiani, D. Kalise, *Invisible Control of Self-Organizing Agents Leaving Unknown Environments*, submitted, 2015.
  - M. Bongini, M. Fornasier, F. Rossi, and F. Solombrino, *Mean-Field Pontryagin Maximum Principle*, submitted, 2015.