Calibration, validation and data assimilation of PDE models in socio-economic sciences[1]

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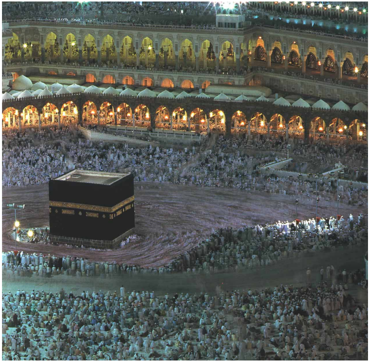
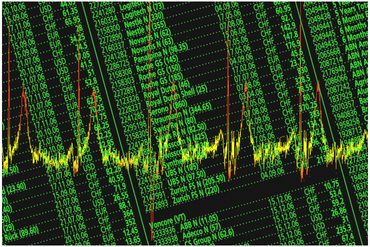
Pedestrian dynamics

- Understanding the flow of large pedestrian crowds and how individual interactions lead to complex macroscopic patters, such as directional lanes, segregation, ...
- But what are the main driving forces in the individual dynamics and how can we determine them from data ?



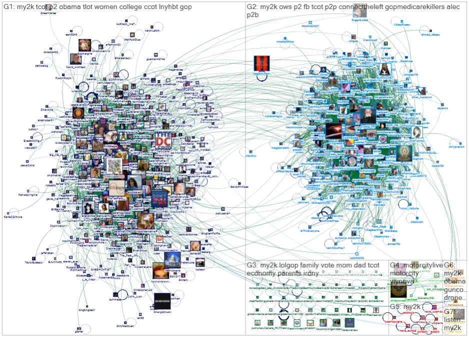
Price formation

- Interplay between trading behaviour and rules, transaction costs, fluctuations in the number of buyers and vendors ..... influences the price dynamics.
- What are minimal features necessary in mathematical models to obtain realistic outcomes ? What is the impact of a single factor ?



Opinion formation, knowledge growth, ...

- Knowledge and information spreads rapidly in social networks like twitter, ...
- How can we use this information to understand consumer choice behaviour, economic growth and make predictions about future dynamics ?



Microscopic models

- The dynamics of each individual and its interactions with all others are described separately.
- Advantage: it's intuitive and rather 'straight forward'.
- Drawback: The more people, the more equations $\Rightarrow$ difficult to predict the large-scale dynamics.

Let $X_i = X_i(t)$ denote a characteristic feature of a single individual at time t, for example his/her position in space, opinion, the price at which he/she is willing to buy/sell a good, ...

Characteristic changes via 'update rules', for example Newton law of motion

$$dX_i = F(X_1, X_2, \dots, X_N)dt + \sigma dW_i(t).$$

or by modelling the one-by-one interactions individuals

$$X_i = X_i + \omega a(X_i, X_j) + \eta_i$$
$$X_j = X_j + \omega a(X_j, X_i) + \eta_j.$$

What are the equations on the macroscopic level ?

PDEs describe the evolution of the overall density with respect to their position and/or velocity in time.
Generic form for the individual density $\varrho = \varrho(x, t)$:

$$\frac{\partial}{\partial t}\varrho(x,t) = \nabla \cdot (m(\varrho)\nabla(\underbrace{U'(\varrho) + W * \varrho + V}_{:=v(\varrho)}),$$

where U corresponds the internal energy, W the interaction energy and V a given external potential.

- Advantage: can be analysed more easily using tools from mathematical analysis.
- Drawback: less intuitive, derivation (from the microscopic level, ad-hoc).

Individual dynamics

Individual based/off-lattice models such as Newton law of motion (known as the 'Social Force Model'):

$$\dot{X}_i(t) = V_i(t)$$
$$\dot{V}_i(t) = F(X_1, \dots, X_N, V_1, \dots, V_N) + G(X_i).$$

Lattice based models (cellular automata, hopping models, ...): State the probability $p_i = p_i(t)$ to find an individual at a discrete lattice site i and update it via

$$\frac{dp_i}{dt} = \mathcal{T}^{+}_{i-1} p_{i-1} + \mathcal{T}^{-}_{i+1} p_{i+1} - (\mathcal{T}^{+}_{i} + \mathcal{T}^{-}_{i}) p_i.$$

Nonlinear PDE models for pedestrian dynamics

Instead of describing the individual dynamics we describe the distribution of individuals.

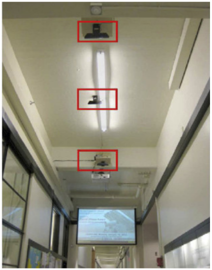
- Boltzmann-type equations for the distribution with respect to position and velocity, that is $f = f(x, v, t)$:

$$\frac{\partial}{\partial t} f(x, v, t) + v \cdot \nabla_x f(x, v, t) + \nabla \cdot (F_f f) = \frac{\sigma^2}{2} \Delta f$$

- Continuity equations for the distribution with respect to their position in space only, for example the Hughes model for pedestrian flow $\varrho = \varrho(x, t)$:

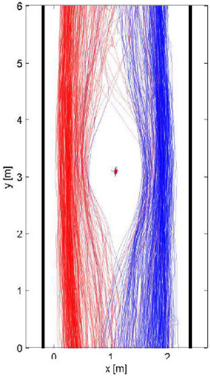
$$\frac{\partial}{\partial t} \varrho(x, t) = \nabla \cdot (\varrho(1 - \varrho)|\nabla \phi|)$$
$$|\nabla \phi|^2 = \frac{1}{1 - \varrho}.$$

Individual trajectories - obtained from cameras[2]



(a) Kinect sensors mounted on the ceiling.

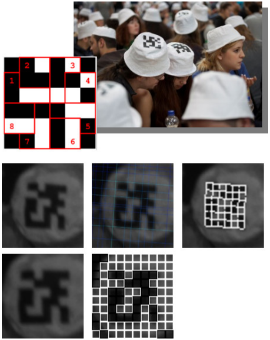
(b) Density map obtained from sensors.



(c) Extracted trajectories.

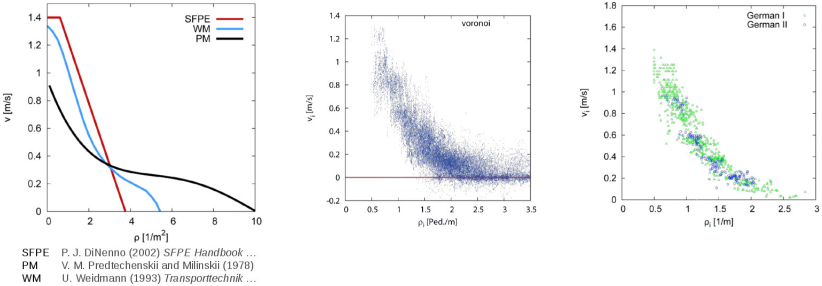
[2] Seer et al., *Validating social force based models with comprehensive real world motion data*, Transportation Research Procedia, 2014

Or from sensors placed on the head ...[3]



[3] Courtesy of Armin Seyfried (Forschungszentrum Jülich), BaSiGo experiments (5 days, 31 experiments, 200 runs, 28 industrial cameras, 2200 participants in total)

Fundamental diagram[4]



(d) Fundamental diagram. (e) Actual measurements. (f) Students vs soldiers.

[4] Courtesy of Armin Seyfried (Forschungszentrum Jülich), BaSiGo experiments (5 days, 31 experiments, 200 runs, 28 industrial cameras, 2200 participants in total)

Calibration and validation in pedestrian dynamics

- Fundamental diagram is commonly used in hyperbolic conservation laws for traffic flow to determine the velocity $v = v(\varrho)$:

$$\frac{\partial}{\partial t}\varrho = \nabla \cdot (\varrho v(\varrho))$$

  $\Rightarrow$ Pedestrians step aside and experimental conditions are not applicable in many conditions.

- Hoogendorn [5] determined model parameters in force based models from experimental trajectories using maximum likelihood estimation.
  $\Rightarrow$ Trajectory data is usually very noisy due to body sway, limited resolution, ...

- Determine mobilities in the nonlinear macroscopic PDE from macroscopic measurements, such as the flow density.
  $\Rightarrow$ Shall we use 'averaged' data to calibrate PDE models ? What do we learn from that ?

[5] Hoogendorn et al., *Microscopic calibration and validation of pedestrian models - Cross comparison using experimental data*, Pedestrian and Evacuation Dynamics, 2005

Individual interactions

Boltzmann-type equations: borrow ideas from statistical physics and model interactions between two individuals via 'collisions'.

- Opinion formation:[6] Toscani suggested the following compromise dynamics if two individuals with opinion X_i and X_j meet:

$$X_i^* = X_i - \omega P(|X_i - X_j|)\,(X_i - X_j) + \eta_i$$
$$X_j^* = X_j - \omega P(|X_i - X_j|)\,(X_j - X_i) + \eta_j$$

- Price dynamics:[7] We consider a buyer willing to buy at price X_i and a vendor willing to sell at price X_j meet. Then the post-interaction prices are determined by the price they agree on $r(X_i, X_j)$ and the transaction costs a.

$$X_i^* = r(X_i, X_j) - a$$
$$X_j^* = r(X_i, X_j) + a$$

[6] Toscani, *Kinetic models of opinion formation*, 2006

[7] Burger, Caffarelli, Markowich and W, *On a Boltzmann type price formation model*, 2013

Individual interactions

Knowledge increase and economic growth[8]:

- Each individual is characterised by its knowledge level $X = X(t)$ and the time $S(t) = S(X(t))$ he/she is willing to spend on learning.
- If two individuals meet (and exchange knowledge), their 'post-collision' knowledge level corresponds to

$$X_i^* = X_j^* = \max(X_i, X_j)$$

- Each individual decides how much time he/she spend on working or learning by maximising:

$$\max_{S(X(t))} \int_{t'}^{T} e^{-r(t-t')}(1 - S(X(t)))X(t)dt.$$

[8] Lucas & Moll, *Knowledge growth and the allocation of time*, 2014

PDE models

Boltzmann-type equation for the distribution of individuals with respect to their position x in space and their velocity v:

$$\frac{\partial}{\partial t} f(x,v,t) - \frac{\sigma^2}{2}\Delta_x f(x,v,t) + v \cdot \nabla_x f(x,v,t) = \mathcal{Q}(f,f)$$

In case of the knowledge growth model this equation is additionally coupled to a Hamilton Jacobi Bellman equation

$$\frac{\partial}{\partial t} V(x,t) - rV(x,t) + \max_s [(1-s(x,t))x - \alpha(s(x,t),t)V(x,t)(1-H)*f+ \\ + \alpha(s(x,t))(1-H)*(V*f)] = 0.$$

PDE models - asymptotic limits

Price formation model for markets with high trading frequency and low transaction costs ⇒ minimal parabolic free boundary problem proposed by Lasry & Lions:

$$\frac{\partial \varrho_b}{\partial t} - \frac{\partial^2 \varrho_b}{\partial x^2} = \lambda(t)\delta\left(x - p(t) + a\right), \quad \text{for } x < p(t)$$

$$\varrho_b(x,t) > 0 \text{ for } x < p(t) \text{ and } \varrho_b(x,t) = 0 \text{ for } x \geq p(t)$$

$$\frac{\partial \varrho_v}{\partial t} - \frac{\partial^2 \varrho_v}{\partial x^2} = \lambda(t)\delta\left(x - p(t) - a\right), \quad \text{for } x > p(t)$$

$$\varrho_v(x,t) > 0 \text{ for } x > p(t) \text{ and } \varrho_v(x,t) = 0 \text{ for } x \leq p(t),$$

with a transaction rate

$$\lambda(t) = -\frac{\partial \varrho_b}{\partial x}\left(p\left(t\right), t\right) = \frac{\partial \varrho_v}{\partial x}\left(p\left(t\right), t\right).$$

GDP, price dynamics, hashtags, ...

- Price formation: Given the price $p = p(t)$ and the transaction rate $\lambda = \lambda(t)$ for a certain time interval $t \in [0, T]$, can we reconstruct the buyer-vendor distribution and predict the price evolution for $t > T$?[9]
- Knowledge growth: Balanced growth path solutions $\Rightarrow$ exponential growth of the overall production
  Knowing that the GDP of most countries grow in the long-term run $\Rightarrow$ what are the implications on the individual interactions or knowledge growth in general ?
- Opinion formation: With more and more data available, such as the dissemination of information or the individual exchange rates on certain topics (for example via Twitter ....)
  $\Rightarrow$ How can we include this information in models and use it to calibrate them ?

[9] J.P. Puel, *A nonstandard approach to a data assimilation problem and Tichonov regularisation revisited*, SICON, 2009

Data assimilation in socio-economic applications

Data assimilation in socio-economic applications
.... matching bad models with even worse data ?!?!

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But to be fair: there is a huge potential, but one has to keep the validity and restrictions in mind.