Convex Relaxations, Semidefinite Optimisation and Applications

Hamza Fawzi

Department of Electrical Engineering and Computer Science, Laboratory for Information and Decision Systems
MIT, Cambridge, MA

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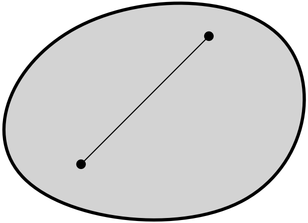
DAMTP / Cantab Capital Institute on Mathematics of Information
Cambridge, UK

May 9th, 2016

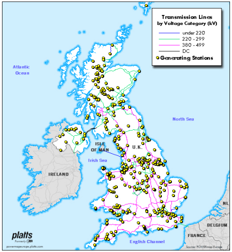
Convexity



- Very rich mathematical theory with applications in many areas
- Fundamental role in *optimisation*
- How to describe convex sets?

A real-world optimisation problem

Optimal power flow



geni.org

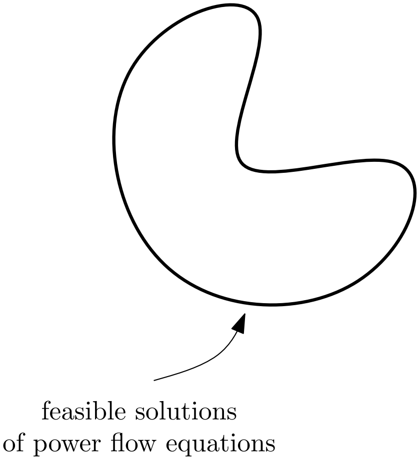
Conservation laws + Ohm's law

$$\textbf{minimise} \quad cost(P)$$

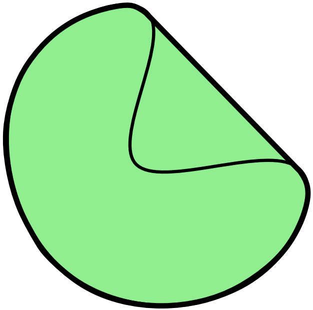
$$\sum_{k \in \mathcal{N}} \frac{1}{R_{ik}} V_i V_k = P_i \quad \text{for each node } i$$

+ constraints on voltage
and power magnitude

Convex formulation



Convex formulation



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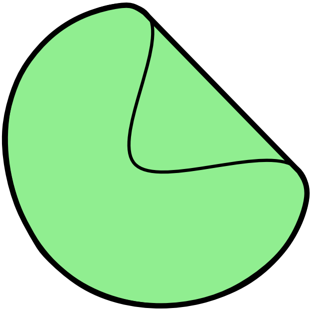
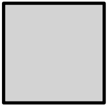


Main question: Can we get an *efficient description* of this convex set?

Describing convex sets

How to describe a convex set?



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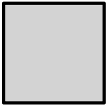

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$\rightarrow$ 4 inequalities

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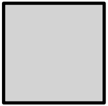

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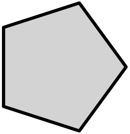

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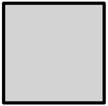

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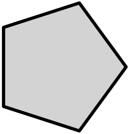

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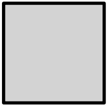

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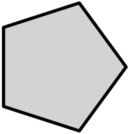

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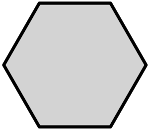

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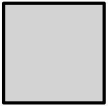

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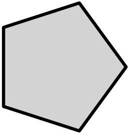

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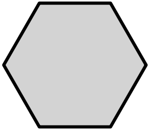

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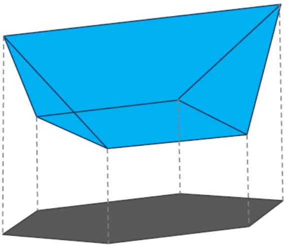
Is there a better way?

Lifting



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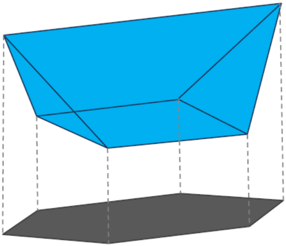
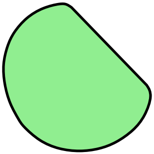


Regular polygon with 2^n sides can be described using only $\approx n$ inequalities!

[Ben-Tal and Nemirovski, 2001]

Nonpolyhedral convex sets



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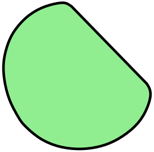

Linear Matrix Inequalities (Semidefinite Optimisation)

$$\begin{bmatrix} 1-x & y \\ y & 1+x \end{bmatrix} \geq 0$$

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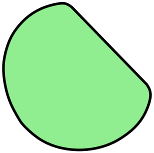

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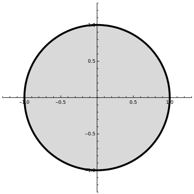

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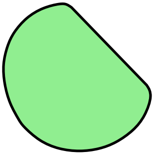

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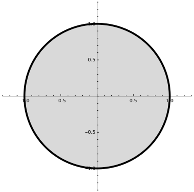

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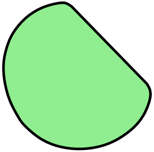

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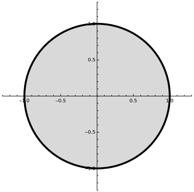

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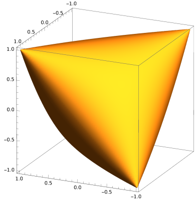

Semidefinite optimisation

- Very powerful framework
- Used in many applications:
  - power flow
  - control theory and dynamical systems
  - combinatorial optimization
  - quantum information theory
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Helton-Nie conjecture: *any convex semialgebraic set is a spectrahedral shadow*

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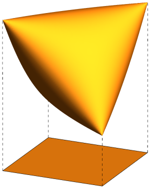

Thank you!