Westfälische Wilhelms-Universität Münster

The p-Laplacian on Graphs with Applications in Image Processing and Classification

Abderrahim Elmoataz[1,2], Matthieu Toutain[1], Daniel Tenbrinck[3]

[1]UMR6072 GREYC, Université de Caen Normandie
[2] Université de Paris-Est Larne-Valée
[3]Department of Computational and Applied Mathematics, WWU Münster

Talk at the Alan Turing Institute
Cambridge, 14.12.2015

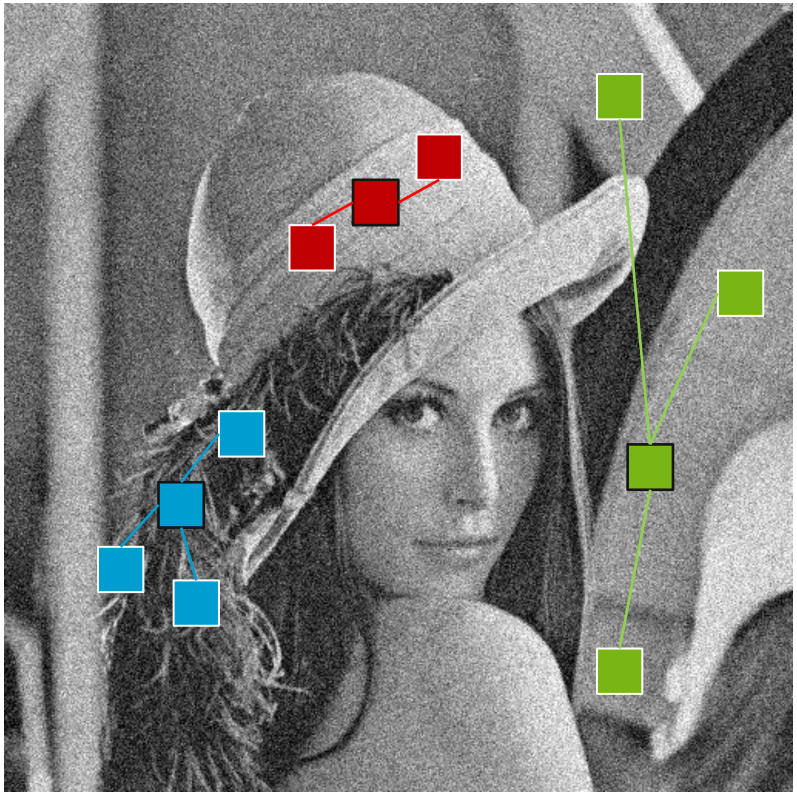
wissen.leben
WWU Münster

Applied Mathematics Münster

1. Motivation
2. Introduction to weighted graphs
3. The graph p-Laplacian operator
4. Solving partial difference equations on graphs

1. Motivation
2. Introduction to weighted graphs
3. The graph p-Laplacian operator
4. Solving partial difference equations on graphs

(Nonlocal) image processing



Aim: Use nonlocal information for image processing
e.g., *denoising, inpainting, segmentation*, …

Surface processing

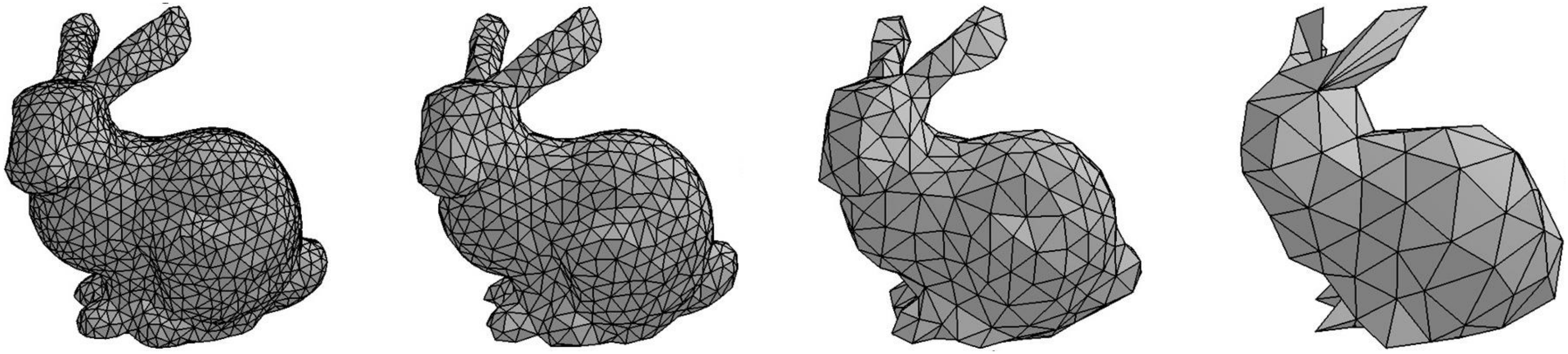
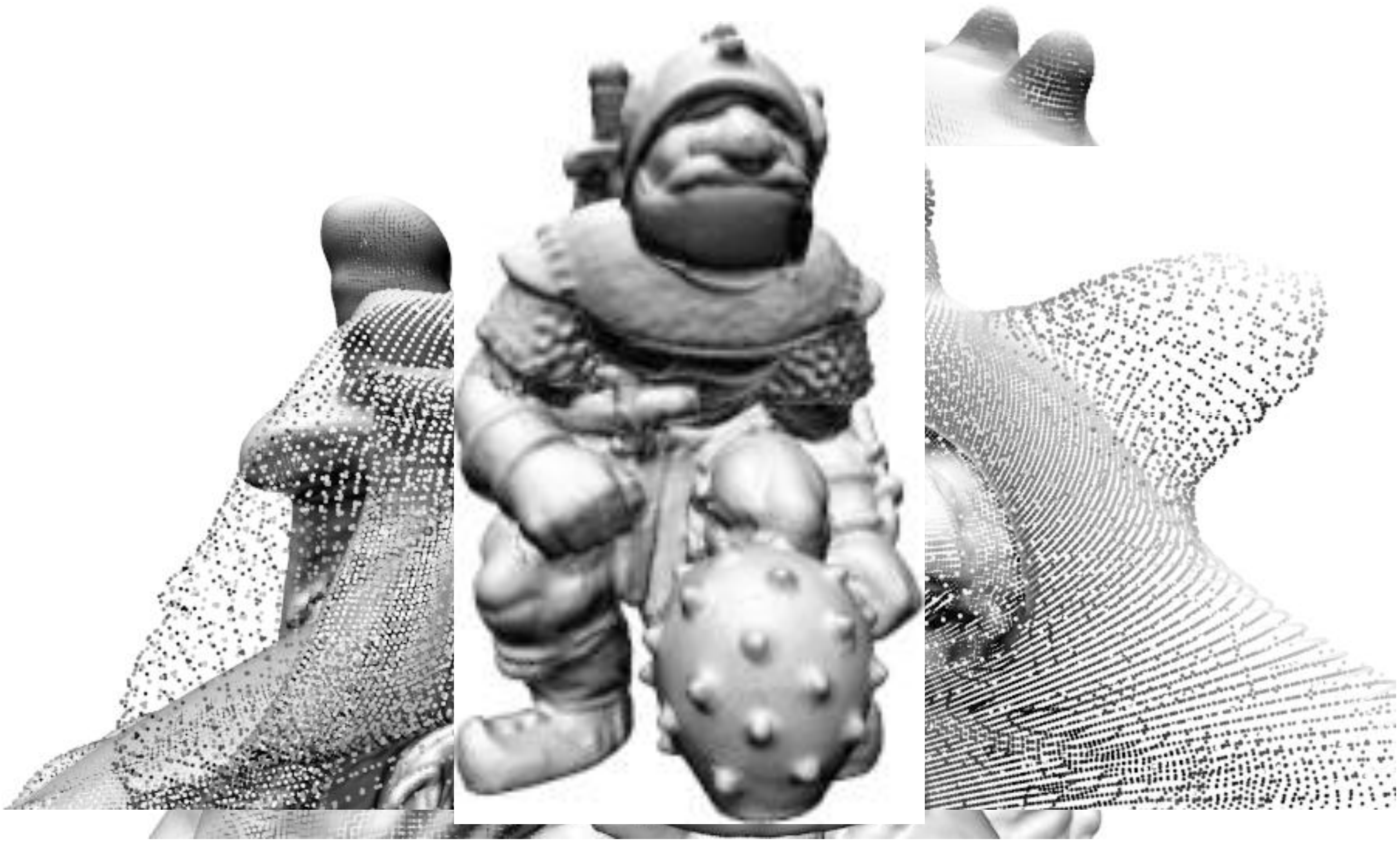


Image courtesy: Gabriel Peyré via http://www.cmap.polytechnique.fr/~peyre/geodesic_computations/

Aim: Approximate surfaces by meshes and process mesh data

e.g., in *computer graphics, finite element methods, …*

Surface processing

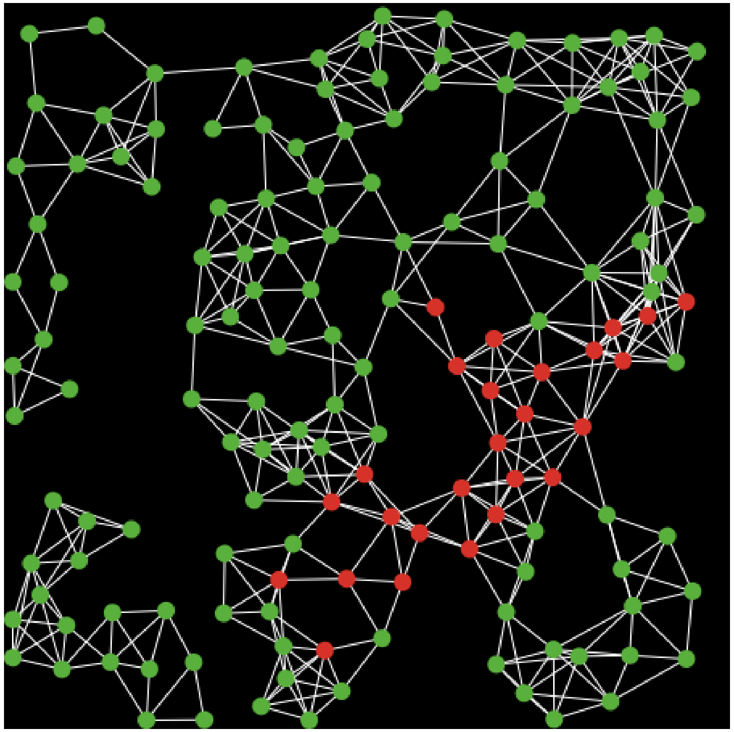


Surface processing

Aim: Approximate surfaces by discrete points and process point cloud data

e.g., in *3D vision, augmented reality, surface reconstruction, …*

Network processing



Aim: Investigate interaction and processes in networks of arbitrary topology

e.g., in *social networks, computer networks, transportation, …*

Graph based methods

Goal:

Use graphs to perform

filtering, segmentation, inpainting, classification, ...

on

data of arbitrary topology.

Question:

How to translate PDEs and variational methods to graphs?

Related works

1. A. Elmoataz, O. Lezoray, S. Bougleux: Nonlocal Discrete Regularization on Weighted Graphs: A Framework for Image and Manifold Processing. IEEE Transactions on Image Processing 17(7) (2008)

2. Y. van Gennip, N. Guillen, B. Osting, A.L. Bertozzi: Mean Curvature, Threshold Dynamics, and Phase Field Theory on Finite Graphs. Milan Journal of Mathematics 82 (2014)

3. F. Lozes, A. Elmoataz, O. Lezoray: Partial Difference Operators on Weighted Graphs for Image Processing on Surfaces and Point Clouds. IEEE Transactions on Image Processing 23 (2014)

4. Leo Grady and Jonathan R. Polimeni, "Discrete Calculus: Applied Analysis on Graphs for Computational Science", Springer (2010)

5. Andrea L. Bertozzi and Arjuna Flenner, Diffuse interface models on graphs for classification of high dimensional data, Multiscale Modeling and Simulation, 10(3), pp. 1090-1118, 2012.

1. Motivation
2. Introduction to weighted graphs
3. The graph p-Laplacian operator
4. Solving partial difference equations on graphs

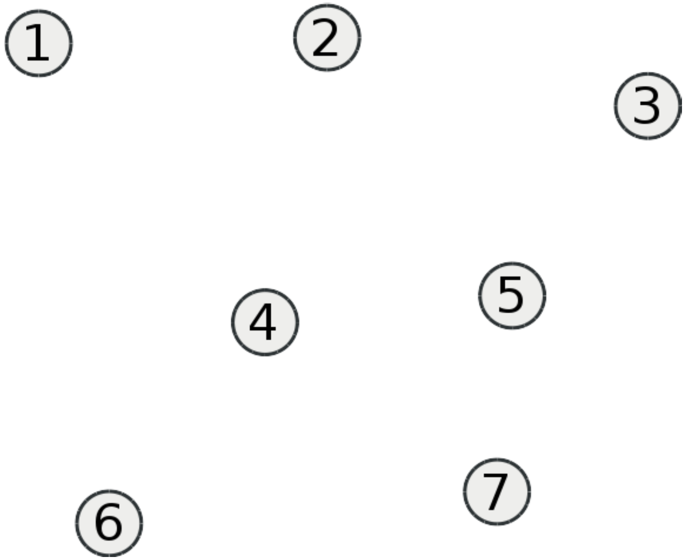
Introducing weighted graphs

A graph G = (V, E, w) consists of:

→ a finite set of *vertices* V = (v_1, …, v_n)

→ a finite set of *edges* E ⊆ V x V

→ a *weight function* w : E → [0, 1]



We consider mainly *undirected*, weighted graphs in the following!

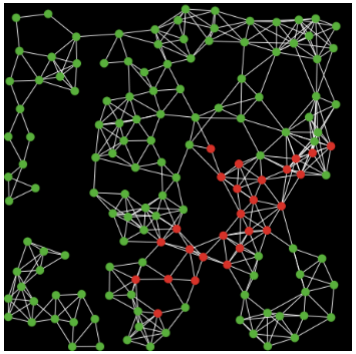
Vertex functions

A vertex function f : V → R^N assigns each v∈V a vector of features:

→ *grayscale value, RGB color vector*

→ *3D coordinates*

→ *label*



The space of vertex functions H(V) is a Hilbert space with the norm:

$$||f|| = \sqrt{\sum_{v_i \in V} \langle f(v_i), f(v_i) \rangle}$$

Weight functions

A weight function w : E → [0, 1] assigns each e∊E a weight based on the similarity of respective node features.

To compute a weight w(u,v) = w(v,u) for nodes u,v ∊ V we need:

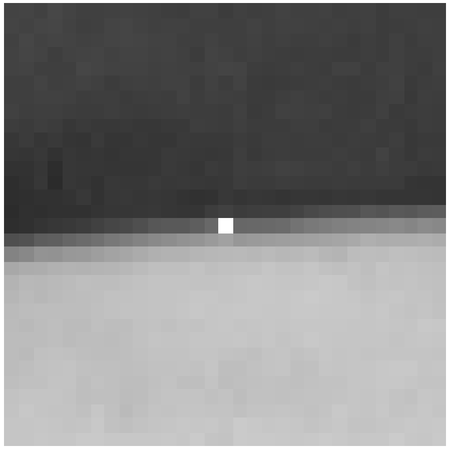
1. Symmetric distance function d(f(u),f(v)) = d(f(v),f(u)) ∊ R
   e.g., constant distance, l^p norms, patch distance, …

2. Normalized similarity function s(d(f(u),f(v))) ∊ [0, 1]
   e.g, constant similarity, probability density function, …

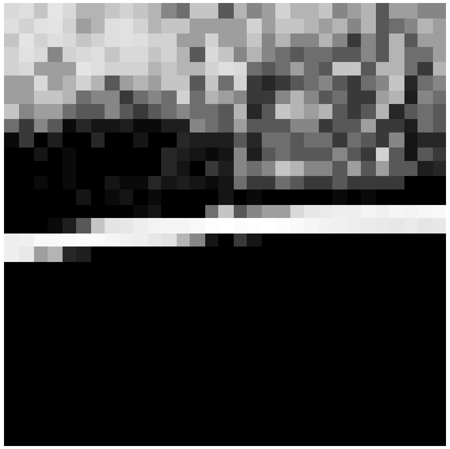
Example:

$$w(u,v) = e^{-\frac{|d(u,v)|^2}{\sigma^2}}, \quad \text{with} \quad d(u,v) = ||p_k(u) - p_k(v)||_2$$

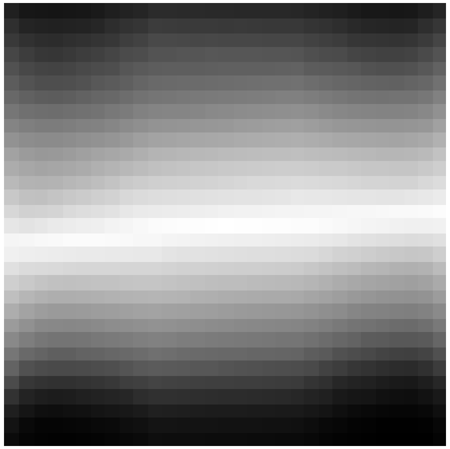
Patch distance



Pixel of interest



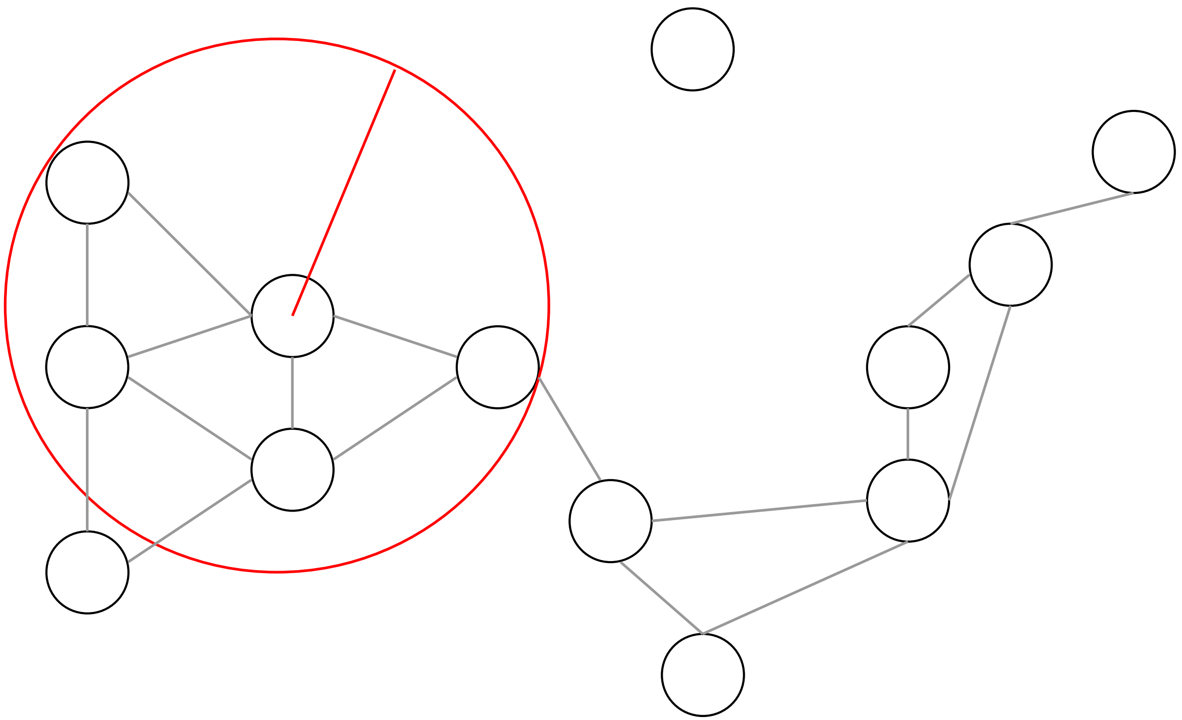
Intensity-based



Patch-based

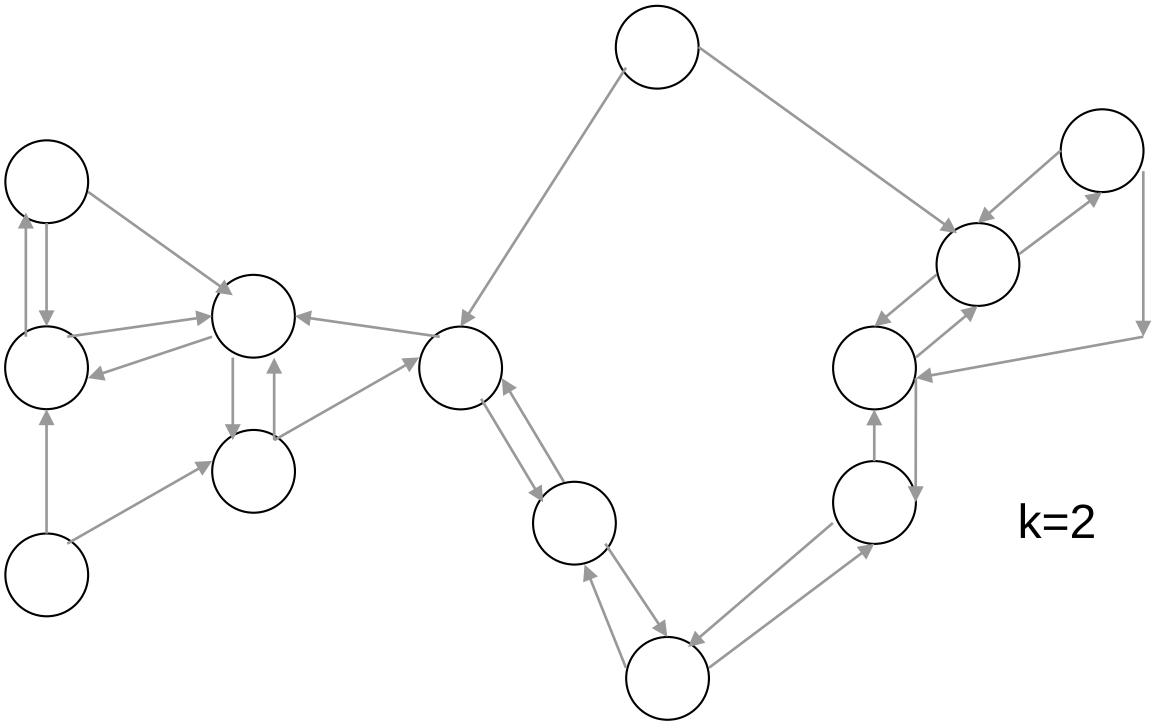
Graph construction methods

1. є-ball graph



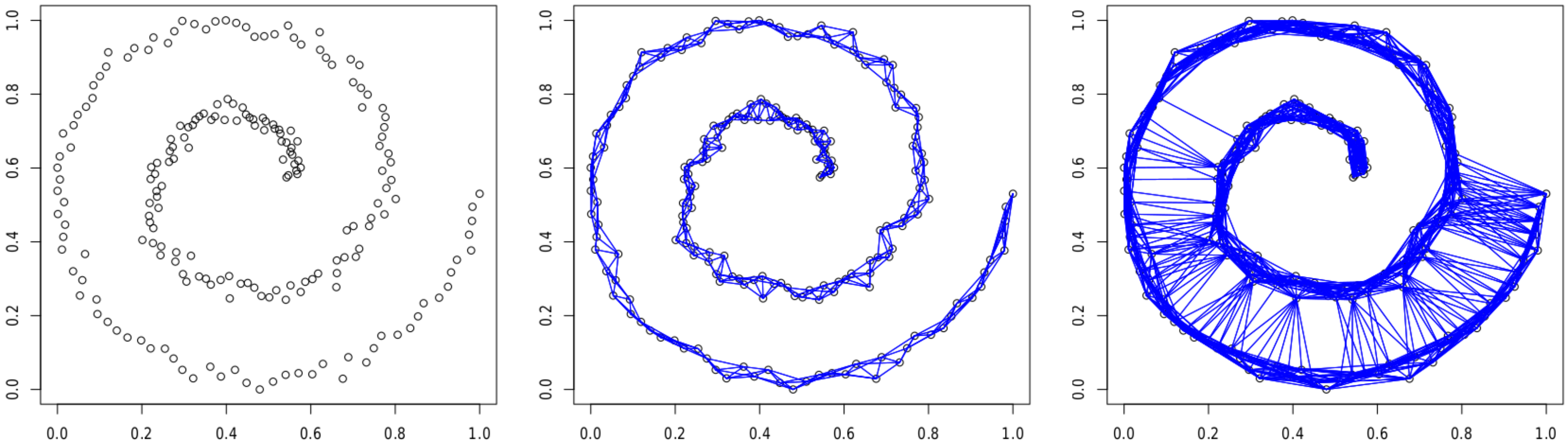
Graph construction methods

2. k-Nearest Neighbour graph (directed):



Graph construction methods

2. k-Nearest Neighbour graph (directed):



(a) (b) k=3 (c) k=15

1. Motivation
2. Introduction to weighted graphs
3. The graph p-Laplacian operator
4. Solving partial difference equations on graphs

Weighted finite differences on graphs

Let (V, E, w) be a weighted graph and let f : V → $\mathbb{R}^N$ be a vertex function. The weighted (nonlocal) finite difference d_w : H(V) → H(E) of f ∊ H(V) along an edge (u,v) ∊ E is given as:

$$d_w f(u,v) = \sqrt{w(u,v)}(f(v) - f(u))$$

Then the weighted gradient of f in a vertex u is given as:

$$(\nabla_w f)(u) = (\partial_v f(u))_{v\in V} \quad \partial_v f(u) = \sqrt{w(u,v)}\,(f(v) - f(u))$$

Similarly, we can introduce the weighted upwind gradient as:

$$\partial_v^{\pm} f(u) = \sqrt{w(u,v)}\big(f(v) - f(u)\big)^{\pm}$$

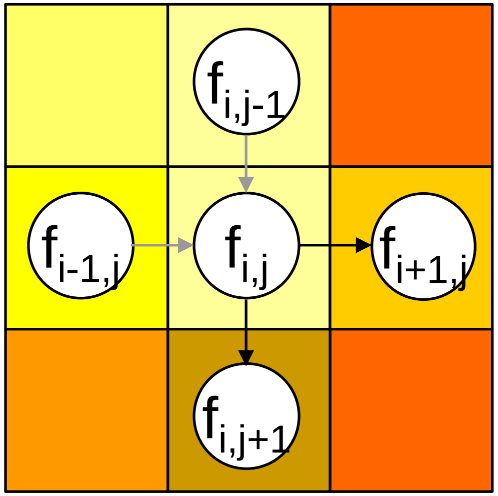
$$(\nabla_w^{\pm} f)(u) = \Big(\partial_v^{\pm} f(u)\Big)_{v\in V}$$

using the notation x^+ = max(0,x) and y^- = -min(0,x).

Special case: grayscale image

Let G = (V, E, w) be a directed 2-neighbour grid graph with the weight function w chosen as:

$$\partial_v f(u) = \sqrt{w(u,v)}\,(f(v) - f(u)) \qquad w(u,v) = \begin{cases} \frac{1}{h^2} & , \quad \text{if } u \sim v \\ 0 & , \quad \text{else} \end{cases}$$



Weighted finite differences correspond to forward differences!

Adjoint operator and divergence

Let f ∊ H(V) be a vertex function and let G ∊ H(E) be an edge function. One can deduce the adjoint operator d^*_w : H(E) → H(V) of d_w : H(V) → H(E) by the following property:

$$\langle d_w f, G \rangle_{\mathcal{H}(E)} = \langle f, d_w^* G \rangle_{\mathcal{H}(V)}$$

Then the divergence div_w : H(E) → H(V) of G in a vertex u is given as:

$$\mathrm{div}_w \, G(u) = -d_w^* G(u) = \sum_{v \sim u} \sqrt{w(u,v)} (G(u,v) - G(v,u))$$

We have in particular the following conservation law:

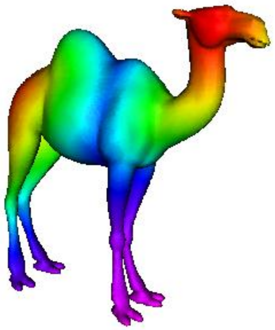
$$\sum_{u \in V} \mathrm{div}_w \, G(u) = 0$$

PdE framework for graphs

We further want to translate PDEs to graphs and formulate them as partial difference equations (PdEs).

Example: Mimic heat equation on graphs

$$\begin{aligned} \frac{\partial f}{\partial t}(u,t) &= \Delta_w f(u,t) \\ f(u,t=0) &= f_0(u) \end{aligned} \quad \text{with} \quad \Delta_w f(u) = \sum_{v \sim u} w(u,v)(f(v) - f(u))$$



The variational p-Laplacian

The variational p-Laplacian is a quasilinear elliptic partial differential operator of second order and can be given for $u \in C^2(\Omega)$ as:

$$\Delta_p u \; = \; \mathrm{div}\left(\frac{\nabla u}{|\nabla u|^{2-p}}\right), \quad 1 \leq p < \infty$$

Note that for $p < 2$ the p-Laplacian has critical points in $\nabla u = 0$.

The p-Laplacian equation with Dirichlet boundary conditions $\Delta_p u \; = \; 0$ can be derived as Euler-Lagrange equation of the minimization problem :

$$E(u) \; = \; \frac{1}{p} \int_\Omega |\nabla u(x)|^p \, \mathrm{d}x$$

Applications:

- Modeling: Biological/physical processes with diffusion, *e.g., heat equation*
- Image processing: total variation (p=1) and Tikhonov regularization (p=2)

The graph p-Laplacian

Let f ∊ H(V) be a vertex function. Then the isotropic graph p-Laplacian operator in an vertex u is given as:

$$(\Delta_{w,p}^{i} f)(u) = \frac{1}{2} \operatorname{div}_w \left(||\nabla_w f||^{p-2} d_w f \right)(u)$$

We can also define the anisotropic graph p-Laplacian operator in an vertex u as:

$$\begin{aligned}(\Delta_{w,p}^{a} f)(u) &= \frac{1}{2} \operatorname{div}_w \left(|d_w f|^{p-2} d_w f \right)(u) \\ &= \sum_{v \sim u} (w(u,v))^{p/2} |f(v) - f(u)|^{p-2} (f(v) - f(u))\end{aligned}$$

1. Motivation
2. Introduction to weighted graphs
3. The graph p-Laplacian operator
4. Solving partial difference equations on graphs

Diffusion processes on graphs

One important class of PdEs on graphs are diffusion processes of the form:

$$\begin{cases} \frac{\partial f(u,t)}{\partial t} & = \Delta_{w,p}^{a} f(u,t) \ , \\ f(u,t=0) & = f_0(u) \ , \end{cases}$$

Applying forward Euler time discretization leads to an iterative scheme:

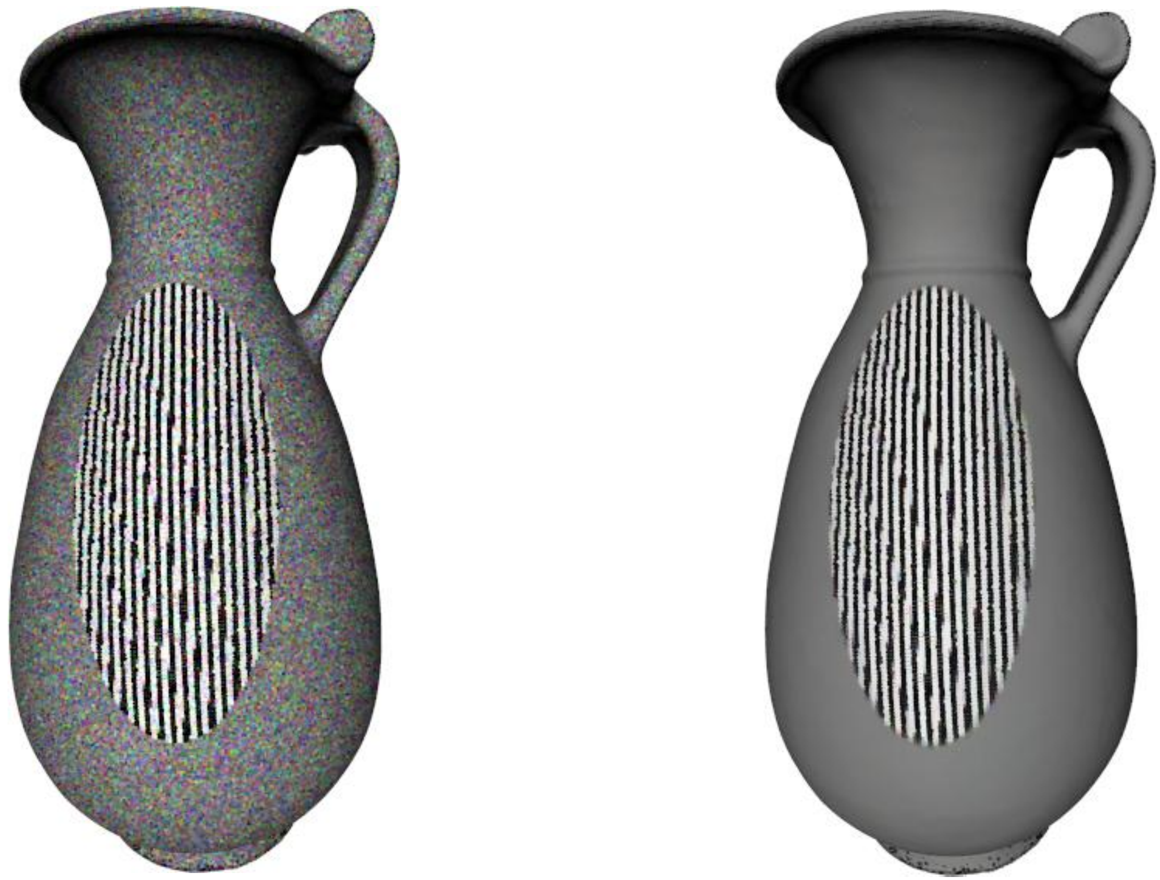
$$f^{n+1}(u) \ = \ f^{n}(u) + \Delta t \sum_{v \sim u} (w(u,v))^{p/2} |f(v) - f(u)|^{p-2} (f(v) - f(u))$$

Maximum norm stability can be guaranteed under the CFL condition:

$$1 \ \geq \ \Delta t \sum_{v \sim u} (w(u,v))^{p/2} |f(v) - f(u)|^{p-2}$$

Denoising

Denoising



Noisy data

Total variation denoising (nonlocal)
1200 iterations

Total variation denoising (local)
600 iterations

Interpolation problems on graphs

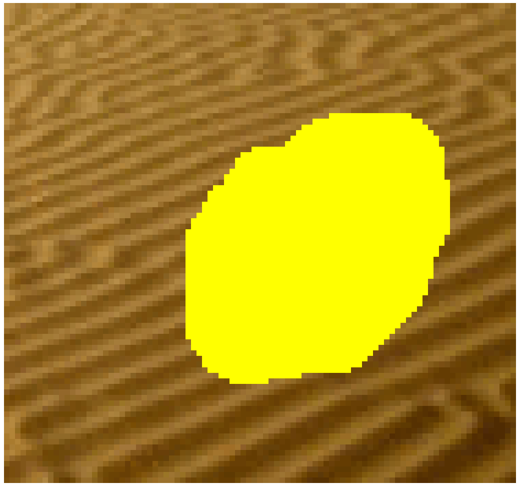
Another class of PdEs on graphs are interpolation problems of the form:

$$\begin{cases} \Delta_{w,p}^{a} f(u) = 0 \,, & \text{for } u \in A \,, \\ f(u) = g(u) \,, & \text{for } u \in \partial A \,. \end{cases}$$

for which A ⊆ V is a subset of vertices and ∂A = V\A and the given information g are application dependent.

Solving this Dirichlet problem amounts in finding the stationary solution of a diffusion process with fixed boundary conditions.

$$\begin{cases} \frac{\partial f(u,t)}{\partial t} = \Delta_{w,p}^{a} f(u,t) \,, & \text{for } u \in A \\ f(u) = g(u) \,, & \text{for } u \in \partial A \,. \end{cases}$$

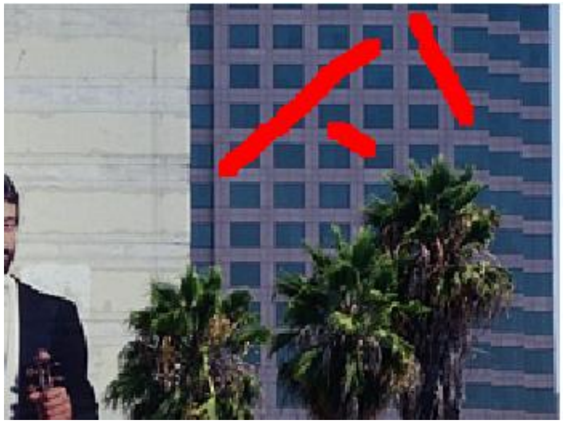


Interactive segmentation

Interactive segmentation

Inpainting


Original image


Inpainting region

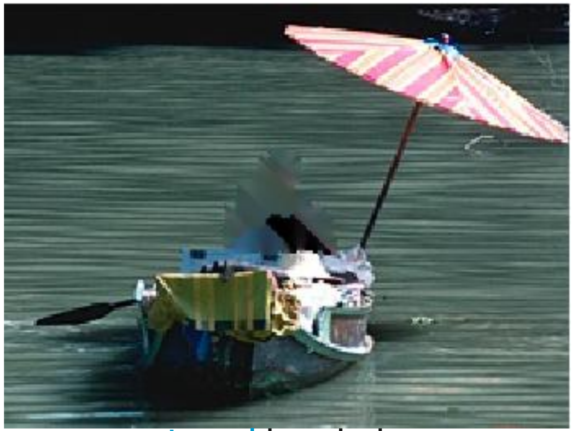

Local inpainting

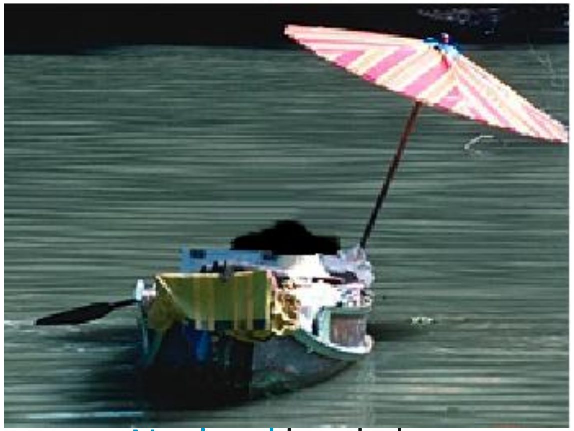

Nonlocal inpainting

Inpainting

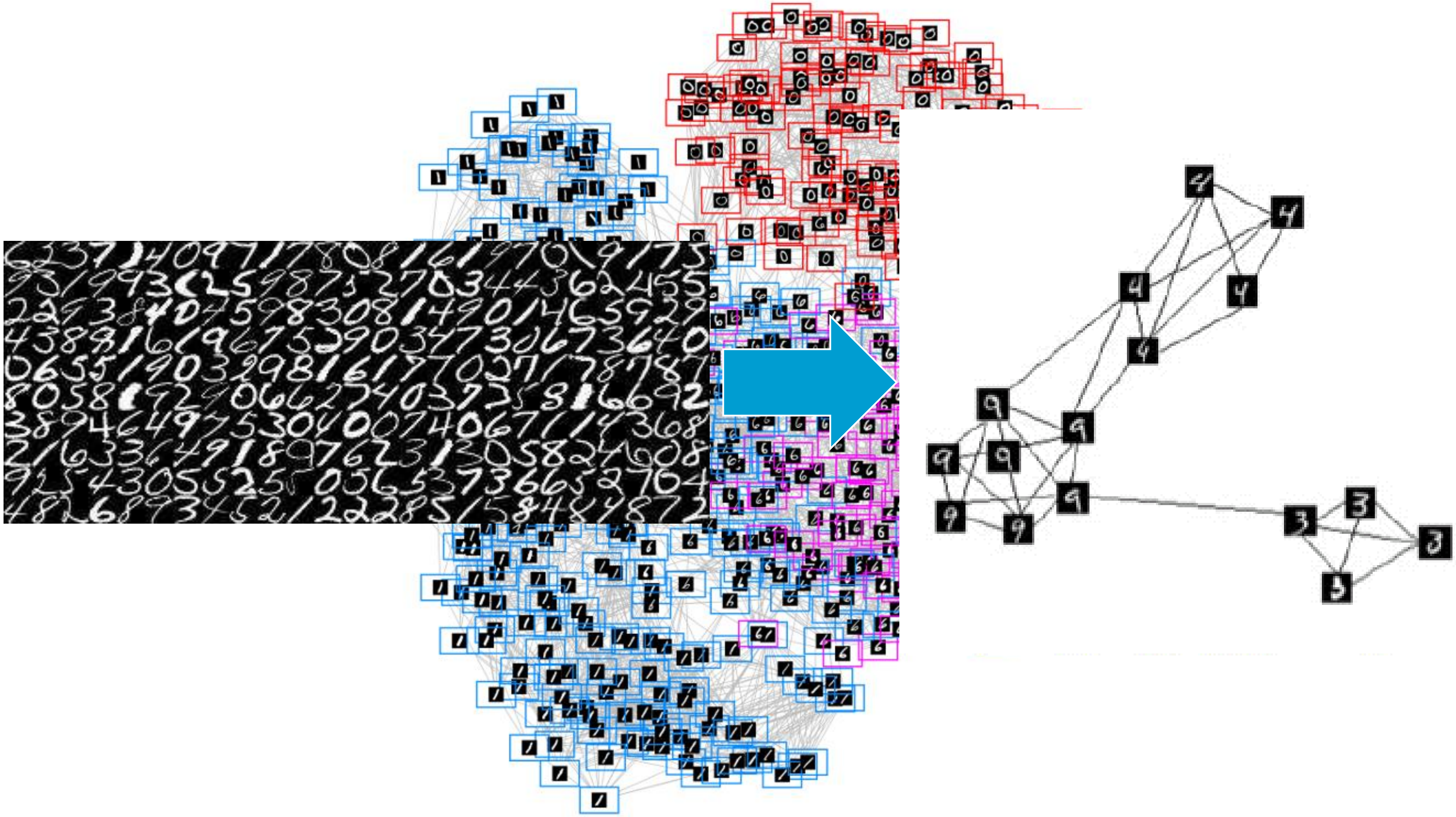

Original image


Inpainting region

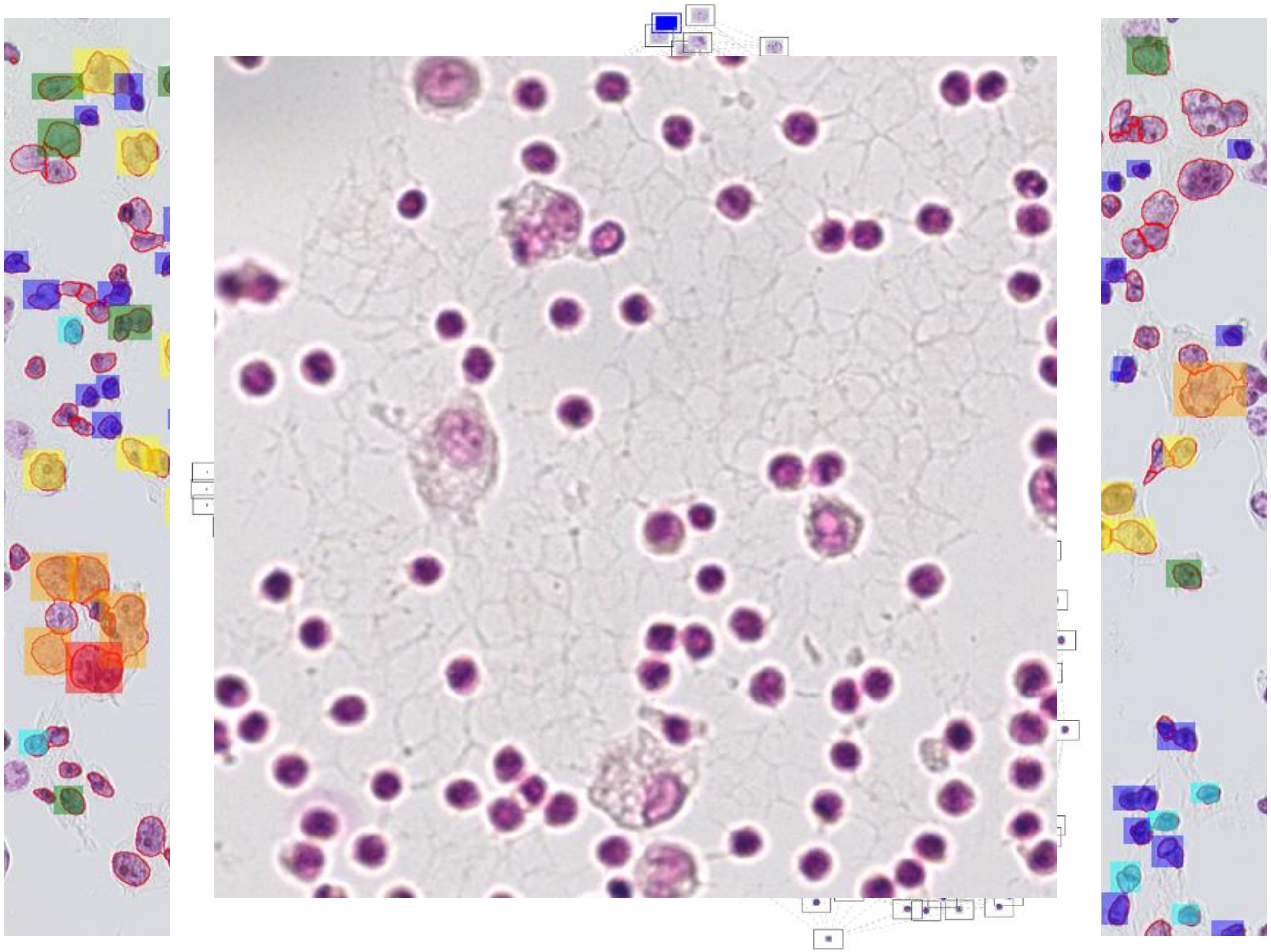

Local inpainting


Nonlocal inpainting

Semi supervised classification



Semi supervised classification



Summary

1) Graph framework unifies local and nonlocal methods

2) PdEs / discrete variational models applicable to data of arbitrary topology

3) Experimental results were demonstrated for:
- *Denoising*
- *Inpainting*
- *Semi supervised segmentation*
- *Classification*

Thank you for your attention! Any questions?

Discrete optimization problems on graphs

We want to mimic variational models on graphs and formulate them as discrete optimization problems.

Example: ROF TV denoising model

$$||u||_{TV,p} = \sum_{x_i \in \mathcal{V}} ||(\nabla_w u)(x_i)||_p = \sum_{x_i \in \mathcal{V}} \left(\sum_{x_j \sim x_i} |(d_w u)(x_i, x_j)|^p \right)^{1/p}, \quad 1 \leq p < \infty$$

$$||u||_{TV,\infty} = \sum_{x_i \in \mathcal{V}} ||(\nabla_w u)(x_i)||_\infty = \sum_{x_i \in \mathcal{V}} \sup_{x_j \sim x_i} |(d_w u)(x_i, x_j)| \quad , \quad p = \infty$$

Find a minimizer $u \in \mathcal{H}(\mathcal{V})$ of the energy

$$E : \mathcal{H}(\mathcal{V}) \to \mathbb{R}, \quad E(u) = \lambda ||u - f||_2^2 + ||u||_{TV}$$

→ Unified formulation for both local and nonlocal problems.

Rudin, Osher, Fatemi:

Variational p-Laplacian and ∞-Laplacian

The variational p- and ∞–Laplacians are quasilinear elliptic partial differential operators of second order and can be given for $u \in C^2(\Omega)$ as:

$$\Delta_p u = \operatorname{div}\left(\frac{\nabla u}{|\nabla u|^{2-p}}\right), \quad 1 \leq p < \infty$$

$$\Delta_\infty u = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_j}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}$$

Note that for $p < 2$ the p-Laplacian has critical points in $\nabla u = 0$.

The p-Laplacian equation with Dirichlet boundary conditions $\Delta_p u = 0$ can be derived as Euler-Lagrange equation of the minimization problem :

$$E(u) = \frac{1}{p}\int_\Omega |\nabla u(x)|^p \, \mathrm{d}x$$

Applications:

- Modeling: Biological/physical processes with diffusion, *e.g., heat equation*
- Image processing: total variation (p=1) and Tikhonov regularization (p=2)

Game p-Laplacian and ∞-Laplacian

The game p- and ∞-Laplacian are also known as normalized Laplacian and they are given as:

$$\Delta_p^G u = \frac{1}{p}|\nabla u|^{2-p}\Delta_p u = \frac{1}{p}|\nabla u|^{2-p}\operatorname{div}\left(\frac{\nabla u}{|\nabla u|^{2-p}}\right), \quad 1 \leq p < \infty$$

$$\Delta_\infty^G u = |\nabla u|^{-2}\Delta_\infty u = |\nabla u|^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 u}{\partial x_i \partial x_j}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}$$

The game p-Laplacian equation $\Delta_p^G u = 0$ is singular whenever $p \neq 2$. One is able to recover the mean curvature flow (p=1) and 2-Laplacian (p=2):

$$\Delta_1^G u = \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)|\nabla u| \qquad \Delta_2^G u = \frac{1}{2}\Delta u$$

The game p-Laplacian can be expressed as convex combination:

$$\Delta_p^G u = \frac{1}{p}\Delta_1^G u + \frac{1}{q}\Delta_\infty^G u, \quad \text{for } \frac{1}{p} + \frac{1}{q} = 1 \text{ and } 1 < p, q < \infty$$

Applications:

- Game theory: Tug-of-War game with noise for p=∞

Nonlocal p-Laplacian

The nonlocal p-Laplacian for a given *continuous*, *normalized*, and *radial* function μ: $\mathbb{R}^n \to \mathbb{R}$ with compact support is given as:

$$\mathcal{L}_p u(x) = \int_\Omega \mu(x-y)|u(y)-u(x)|^{p-2}(u(y)-u(x))\,\mathrm{d}y\ , \quad 1 \leq p < \infty$$

If the convolution kernel μ is chosen as $\mu(x-y) = \frac{1}{|x-y|^{n+ps}}\ , \quad p \geq 1,\ 0 < s < 1$ one derives the fractional p-Laplacian:

$$\mathcal{L}_p u(x) = \int_\Omega \frac{|u(y)-u(x)|^{p-2}}{|x-y|^{n+ps}}(u(y)-u(x))\,\mathrm{d}y\ , \quad 1 \leq p < \infty$$

In the case p=∞ this operator corresponds to the Hölder ∞-Laplacian:

$$\mathcal{L}_\infty u(x) = \max_{y\in\Omega, y\neq x}\left(\frac{u(y)-u(x)}{|y-x|^\alpha}\right) + \min_{y\in\Omega, y\neq x}\left(\frac{u(y)-u(x)}{|y-x|^\alpha}\right)$$

Applications:

- Image processing: Nonlocal regularization
- Modeling: Quantum phenomena in physics or population dynamics

Chambolle, Lindgren, Monneau: A Hölder infinity Laplacian. ESAIM: Control, Optimisation and Calculus of Variations 18 (2012)

A novel Laplacian operator on graphs

Let G = (V, E, w) be a weighted undirected graph and α, β : H(V) →$\mathbb{R}^N$ vertex functions with α(u) + β(u) = 1 for all u∈V. We propose a novel Laplacian operator on G as:

$$\mathcal{L}_{\omega,p} f(u) = \begin{cases} \alpha(u)\|(\nabla_w^+ f)(u)\|_{p-1}^{p-1} - \beta(u)\|(\nabla_w^- f)(u)\|_{p-1}^{p-1}, & 2 \leq p < \infty \\ \alpha(u)\|(\nabla_w^+ f)(u)\|_\infty - \beta(u)\|(\nabla_w^- f)(u)\|_\infty, & p = \infty \end{cases}$$

From previous works it gets clear that:

$$\Delta_{w,p} f(u) = ||\nabla_w^+ f(u)||_{p-1}^{p-1} - ||\nabla_w^- f(u)||_{p-1}^{p-1}$$

A simple factorization leads to this representation:

$$\begin{aligned} \mathcal{L}_{w,p} f(u) = 2\min(\alpha(u), \beta(u))\Delta_{w,p} f(u) \\ + (\alpha(u) - \beta(u))^+ \|(\nabla_w^+ f)(u)\|_{p-1}^{p-1} \\ - (\alpha(u) - \beta(u))^- \|(\nabla_w^- f)(u)\|_{p-1}^{p-1}, \quad 2 \leq p < \infty \end{aligned}$$

$$\begin{aligned} \mathcal{L}_{w,\infty} f(u) = 2\min(\alpha(u), \beta(u))\Delta_{w,\infty} f(u) \\ + (\alpha(u) - \beta(u))^+ \|(\nabla_w^+ f)(u)\|_\infty \\ - (\alpha(u) - \beta(u))^- \|(\nabla_w^- f)(u)\|_\infty \end{aligned}$$

Observation:

Novel operator is a combination of p-Laplacian and upwind gradient operators on graphs