Kinetic equations modelling complex systems in economics and sociology

Bertram Düring

Department of Mathematics

University of Sussex

Joint work with

P. Markowich, D. Matthes, J.-F. Pietschmann,
G. Toscani, M.-T. Wolfram

Wealth distribution in an economy

Wealth: total value of a person's net assets (money, shares, property, buildings, . . .) from which any debts are subtracted

Exhibit: Income distribution in the USA

- upper class ($\approx 1\%$) well fitted by inverse power law (Pareto tail)
- important implications for taxation, social peace, growth

Opinion formation

- How are political opinions formed?
- Sociology tells us that interpersonal communication and opinion leaders play a big role

- not limited to politics, but more general decision making: product choice, dissemination of new technologies,...

Complex systems in economics and sociology

Examples:

- distribution of wealth in a society
- opinion formation (political opinions, product choice, ...)

Common features:

- large number of interacting agents (agent-based models?)
- model of full system not tractable
- quantities of interest are aggregates
- dynamics!
- emergent behaviour

$\rightsquigarrow$ mathematical tools from kinetic theory/statistical mechanics

From micro to macro

Conceptual approach:

- continuous independent variables:
  wealth/price/opinion and time
- describe dynamics of system by microscopic interactions among agents
- perform many interactions (analytically or numerically)
- observe emergent behaviour, patterns in macroscopic distribution of agents
- derive partial differential equations which (approximatively) govern the time-evolution of the density

Benefits:

- more (analytically and numerically) tractable model
- understanding role of parameters in the microscopic interactions for emergent behaviour
- PDE: nonlinear, anisotropic, nonlocal, degenerate

Wealth distribution: quantity of interest

- We are interested in the large-wealth behavior of the distribution since it determines *a posteriori* if the model results fits real data
- let $f(v)$ be probability density function of agents with wealth v, and consider the commulative density

$$F(w) = \int_{w}^{\infty} f(v)\, dv$$

- over 100 years ago, the Italian economist Vilfredo Pareto first quantified the large-wealth behavior to follow a power-law distribution
- i.e. $F(w) \sim w^{-\mu}$ for w large

Kinetic models of simple markets

Idea: model agents in simple market as colliding molecules in a Boltzmann gas:

Economic system	Particle dynamics
agents	molecules
agent's wealth	particle momentum
mean wealth	total momentum
transactions	collisions

However, complete analogy fails:

- no debts allowed $\longrightarrow$ particle momenta non-negative
- risky investments $\longrightarrow$ collision kernel with randomness
- individual agent's preferences $\longrightarrow$ particles distinguishable

Goals

- kinetic model of a simple economy to describe evolution of the *macroscopic* wealth distribution by means of *microscopic* interactions (trades) among individuals (agents)
- we will consider binary interactions of the form

$$v^* = p_1 v + q_1 w, \qquad w^* = p_2 v + q_2 w$$

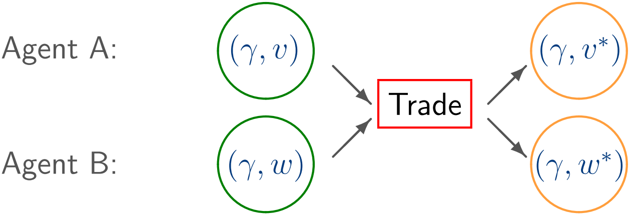
- wealth after trade non-negative (no debts)$\rightarrow v^*, w^* \geq 0$
- p_i, q_i can be given constants or random quantities
- p_i, q_i should be nonnegative

Benchmark: steady states of good model should show Pareto tail

Model of Cordier, Pareschi & Toscani (2005)

→ w/w^*, v/v^* denote agents' wealths before/after trade

→ single, fixed saving propensity $\gamma \in (0, 1)$



→ binary trade is characterized by

$$v^* = \gamma v + (1-\gamma)w + \eta v$$
$$w^* = \gamma w + (1-\gamma)v + \tilde{\eta} w$$

where risks of the market are described by η and $\tilde{\eta}$, random variables with same distribution and zero mean

Boltzmann-type equation

Paradigm (∼1985)

Wealth distribution function $f(w,t)$ *satisfies a homogeneous Boltzmann equation on* $\mathbb{R}_+$

$$\partial_t f = \mathcal{Q}(f,f), \quad f(w,0) = f_0(w).$$

Weak form: For all smooth test functions ϕ consider

$$\frac{d}{dt}\int_{\mathbb{R}_+} f(w,t)\phi(w)\,dw = \int_{\mathbb{R}_+} \mathcal{Q}(f,f)\phi(w)\,dw,$$

where

$$\int_{\mathbb{R}_+} \mathcal{Q}(f,f)\phi(w)\,dw = \frac{1}{2}\Big\langle \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} [\phi(w^*)+\phi(v^*)-\phi(w)-\phi(v)] f(w)f(v)\,dwdv \Big\rangle$$

Interpretation: Pre-trade v, w change into post-trade v^*, w^*

→ to define a specific model, prescribe transitions $(v,w) \rightsquigarrow (v^*,w^*)$

Boltzmann equation: Fourier version

Existence of solutions to Boltzmann equation follows from results for the elastic Kac model

Idea: Boltzmann-like equation with constant kernel easily studied using *Fourier transform* (Bobylev, 1988)

Fourier version

$$\frac{\partial \widehat{f}(t;\xi)}{\partial t} = \widehat{Q}\big(\widehat{f},\widehat{f}\big)(t;\xi)$$

can be written as

$$\frac{\partial \widehat{f}(t;\xi)}{\partial t} = \frac{1}{2}\big\langle \widehat{f}(p_1\xi)\widehat{f}(q_1\xi) + \widehat{f}(p_2\xi)\widehat{f}(q_2\xi)\big\rangle - \widehat{f}(t;\xi)$$

Unified approach

[B.D./Matthes/Toscani '08, Matthes/Toscani '08]

Use Fourier metric

$$d_s[f_1, f_2] = \sup_\xi [|\xi|^{-s} |\widehat{f_1}(\xi) - \widehat{f_2}(\xi)|], \quad s > 0$$

Theorem

Let $f_1(t)$ and $f_2(t)$ be two solutions of the Boltzmann equation, corresponding to initial values with same mean.
Let $s \geq 1$ be such that $d_s[f_{1,0}, f_{2,0}]$ is finite.
Then for all $t \geq 0$

$$d_s[f_1(t), f_2(t)] \leq \exp\left[\left(\frac{1}{2}\Big(\sum_{i=1}^{2} \langle p_i^s + q_i^s \rangle\Big) - 1\right) t\right] d_s[f_{1,0}, f_{2,0}].$$

Characteristic function

Introduce the characteristic function

$$\mathcal{S}(s) = \frac{1}{2}\Big(\sum_{i=1}^{2}\langle p_i^s + q_i^s\rangle\Big) - 1$$

- obviously $\mathcal{S}(0) = 1$
- $\mathcal{S}(1) = 0$ because of the conservation property
- sign of $\mathcal{S}(s)$ is related to number of moments of solution which remain uniformly bounded in time
- Large-time behavior of solution depends on sign of $\mathcal{S}(s)$

Conclusions for CPT (2005) model

Mixing parameters are given by

$$p_1 = \gamma + \eta, \qquad q_1 = 1 - \gamma$$
$$p_2 = 1 - \gamma, \qquad q_2 = \gamma + \tilde{\eta}$$

This implies conservation in the mean, i.e. $\langle v^* + w^* \rangle = v + w$.

Need to evaluate

$$\mathcal{S}(s) = (1-\gamma)^s - 1 + \tfrac{1}{2}\langle (\gamma+\eta)^s + (\gamma+\tilde{\eta})^s \rangle$$

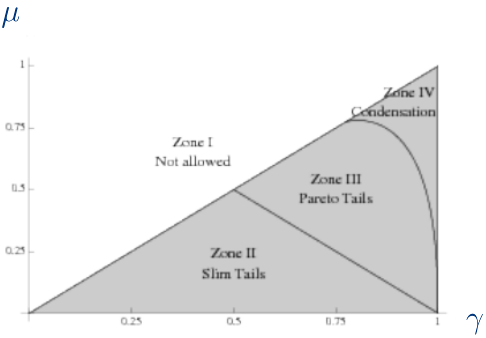
Conclusions for CPT model: specific example

- Let η only assume values $\pm\mu$, with probability $\frac{1}{2}$ each, where $0 \leq \mu \leq \gamma$
- by varying γ and μ one encounters variety of possible outcomes
- if $\gamma + \mu \leq 1$, trade is pointwise conservative
  $\rightarrow$ all moments are finite (exponential tail)
- if $\gamma + \mu > 1$, evaluate

$$\mathcal{S}(s) = (1-\gamma)^s - 1 + \frac{1}{2}(\gamma+\mu)^s + \frac{1}{2}(\gamma-\mu)^s$$

Conclusions for CPT model: specific example

One obtains the following classification for f_∞

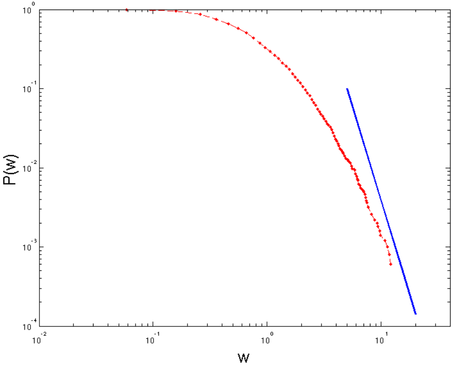


- Zone I: exluded by $\mu \leq \gamma$
- Zone II: Socialism (exponential tails)
- Zone III: Capitalism (Pareto tails)
- Zone IV: Plutocracy (Dirac distribution)

Numerical example for CPT model

Kinetic Monte Carlo simulation with $N = 5000$ agents

- each agent starts with unit wealth
- uniform saving propensity $\gamma \equiv 0.7$ and $\eta = \pm 0.5$
- agents are randomly selected for trade events
- one time step = N collisions



▷ Pareto tail $w^{-\bar{s}}$,
▷ $\bar{s} \approx 3.7$ $(\mathcal{S}(\bar{s}) = 0)$

Kinetic models for opinion formation: literature

- Toscani (2006): opinion variable $w \in \mathcal{I} = [-1, 1]$
- Boudin/Salvarani (2008): opinion variable $w \in \mathcal{I}^p = [-1, 1]^p$
- Bertotti/Delitala (2009): opinion leadership, discrete model
- B.D./Markowich/Pietschmann/Wolfram (2009): opinion leadership, two species (PRSA)
- Motsch/Tadmor (2014), Heterophilious dynamics enhances consensus (SIAM Review)
- Pareschi/Toscani (2014): Book 'Interacting Multiagent Systems'
- ...

Connections to swarming/flocking models (Barbaro, Bertozzi, Cañizo, Carrillo, D'Orsogna, Slepčev,...) and 'sociophysics' literature (Galam *et al.*, Hegselmann/Krause,...)

Opinion formation: Toscani (2006) model

- describe evolution of opinion distribution by microscopic interactions among individuals
- society develops a certain macroscopic opinion distribution
- opinion: continuous variable $w \in \mathcal{I} = [-1, 1]$, where ± 1 represent extreme opinions

Two individuals with pre-interaction opinion v and w meet $\rightarrow$ post-interaction opinions v^* and w^* are given by

$$v^* = v - \gamma P(|v - w|)(v - w) + \eta_1 D(v),$$
$$w^* = w - \gamma P(|w - v|)(w - v) + \eta_2 D(w).$$

$\gamma \in (0, \frac{1}{2})$: constant compromise parameter
$\eta_{1,2}$: random variables with mean zero and variance σ^2 modeling self-thinking through an exogenous, global access to information, e.g. through the press, television or internet.

Homogeneous Boltzmann-like equation

Homogeneous society [Toscani '06]
$\rightsquigarrow$ Boltzmann-like equation for the opinion distribution function $f = f(w,t)$

$$\frac{\partial}{\partial t} f(w,t) = \frac{1}{\tau} \mathcal{Q}(f,f)(w)$$

τ : relaxation time
Collision operator in weak form:

$$\int_{\mathcal{I}} \mathcal{Q}(f,f)(w) \phi(w)\, dw$$
$$= \frac{1}{2} \left\langle \int_{\mathcal{I}^2} \big(\phi(w^*) + \phi(v^*) - \phi(w) - \phi(v)\big) f(v) f(w)\, dv\, dw \right\rangle$$

for all smooth functions $\phi(w)$

Opinion leadership in sociology

- sociological concept trying to explain formation of opinions in a society
- going back to Lazarsfeld *et al.* (1944) studying US presidential elections 1940
- found out interpersonal communication to be much more influential than direct media effects
- two-step flow of communication: opinion leaders (active media users) select, interpret, modify, facilitate, and finally transmit information to less active parts of the population

Opinion leader characteristics

Typical opinion leader characteristics are

- high confidence
- high self-esteem
- strong need to be unique (public individuation)
- socially active, highly connected (scale-free network)
- ability to withstand criticism

Although different, not easy to distinguish from followers

- still related to their followers
- opinion leadership is specific to a subject and can change over time

Kinetic model with opinion leaders

[B.D./Markowich/Pietschmann/Wolfram, '09]

Inhomogeneous society consisting of two groups

- opinion leaders (highly self-confident, assertive, able to withstand criticism)
- followers

If two individuals from the same group meet $(i = 1, 2)$

$$v^* = v - \gamma_i P_i(|v - w|)(v - w) + \eta_{i1} D_i(v)$$
$$w^* = w - \gamma_i P_i(|w - v|)(w - v) + \eta_{i2} D_i(w)$$

Follower with opinion v meets opinion leader with opinion w

$$v^* = v - \gamma_3 P_3(|v - w|)(v - w) + \eta_{11} D_1(v)$$
$$w^* = w$$

$\gamma_k \in (0, \frac{1}{2})$: compromise parameters, which control the 'speed' of attraction of two different opinions

η_{ij} : random variables with variance σ_{ij}^2 and zero mean

Boltzmann system

Distribution functions $f_i = f_i(w,t)$ obey system of two Boltzmann-like equations, given by

$$\frac{\partial}{\partial t} f_i(w,t) = \sum_{j=1}^{n} \frac{1}{\tau_{ij}} \mathcal{Q}_{ij}(f_i, f_j)(w).$$

τ_{ij} : relaxation times

Collision operators in weak form:

$$\int_{\mathcal{I}} \mathcal{Q}_{ij}(f_i, f_j)(w)\phi(w)\, dw$$
$$= \frac{1}{2} \left\langle \int_{\mathcal{I}^2} \big(\phi(w^*) + \phi(v^*) - \phi(w) - \phi(v)\big) f_i(v) f_j(w)\, dv\, dw \right\rangle$$

for all smooth functions $\phi(w)$

Fokker-Planck system

Idea: Derive macroscopic approximation

For $\gamma \ll 1$, $\tau = \gamma t$, let $g_i(w,\tau) = f_i(w,t)$.

Limit: $\gamma, \sigma_{ij} \to 0$ with $\lambda_{ij} = \sigma_{ij}^2/\gamma$ fixed

$\to$ (scaled) densities converge to $g_i(w,\tau)$, which solve

$$\frac{\partial}{\partial \tau} g_1(w,\tau) = \frac{\partial}{\partial w}\left(\left(\frac{1}{\tau_{11}}\mathcal{K}_1(w,\tau) + \frac{1}{2\tau_{12}}\mathcal{K}_3(w,\tau)\right) g_1(w,\tau)\right) + \left(\frac{\lambda_{11}M_1}{2\tau_{11}} + \frac{\lambda_{12}M_2}{4\tau_{12}}\right)\frac{\partial^2}{\partial w^2}\left(D_1^2(w) g_1(w,\tau)\right)$$

$$\frac{\partial}{\partial \tau} g_2(w,\tau) = \frac{\partial}{\partial w}\left(\frac{1}{\tau_{22}}\mathcal{K}_2(w,\tau) g_2(w,\tau)\right) + \frac{\lambda_{22}M_2}{2\tau_{22}}\frac{\partial^2}{\partial w^2}\left(D_2^2(w) g_2(w,\tau)\right)$$

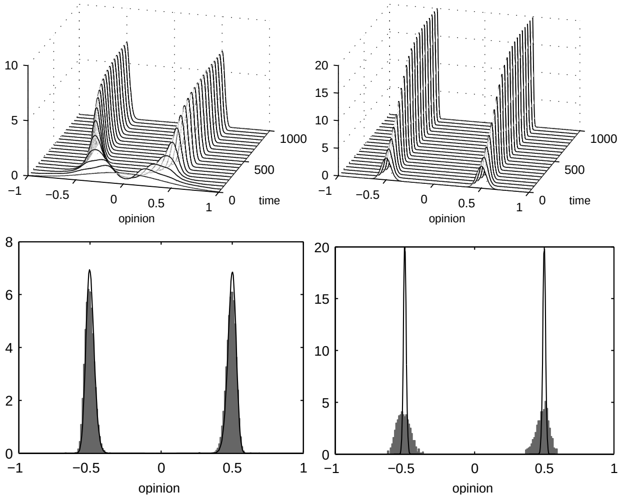
with nonlocal drift operators

$$\mathcal{K}_i(w,\tau) = \int_{\mathcal{I}} P_i(|w-v|)(w-v) g_i(v,\tau)\, dv \quad \text{for } i = 1,2$$

$$\mathcal{K}_3(w,\tau) = \int_{\mathcal{I}} P_3(|w-v|)(w-v) g_2(v,\tau)\, dv$$

Numerical results

$$P_i(|v-w|) = \mathbf{1}_{\{|v-w|\le r_i\}},\ r_i = 0.5$$
$$D_1(w) = D_2(w) = D(w) := (1-w^2)^2$$



Understanding Carinthia

- Carinthia is the southernmost state of Austria
- since 1999 elections the right-wing Freedom Party of Austria (FPÖ) became strongest party and continually improved, holding almost 45 % of the votes in 2008

- outcome strongly influenced by popularity of party leader Haider
- being considered populistic, extreme-right or even antisemitic by many
- strongly acclaimed by his followers
- success less founded on political ideas than on authority of Haider himself

Example: Carinthia state elections

Let's illustrate the behavior of our model under extreme conditions, like in Carinthia.

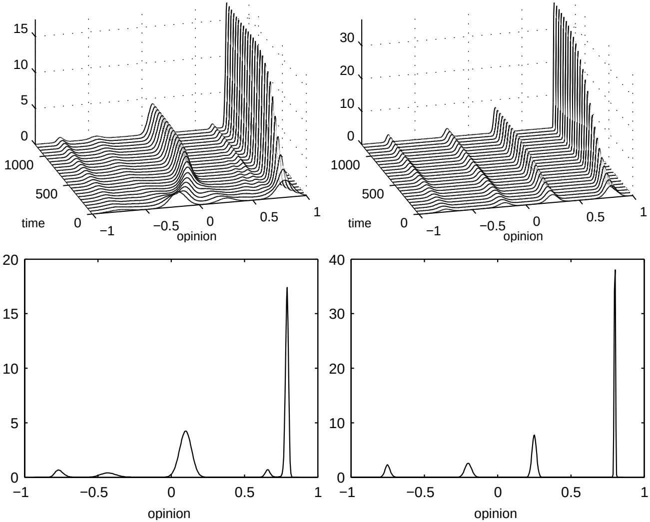
Table: Results of the state elections in Carinthia

	Grüne	SPÖ	ÖVP	FPÖ	BZÖ
2004	6.7%	38.4 %	11.6 %	42.5 %	—
2009	5.2%	28.8 %	16.8 %	3.8 %	44.9 %

- initial distribution of normal people: weighted sum of Gaussians according to 2004 elections
- opinion leaders associated with the different parties with weights according to their influence
- other parameters

$$\alpha = 1.5, \quad \lambda = 3 \times 10^{-3}, \quad r_1 = r_2 = 0.2, \quad r_3 = 0.45,$$
$$\tau_{11} = \tau_{12} = 1, \quad \tau_{22} = 10, \quad \sigma_1 = 0.1, \quad \sigma_2 = 0.05.$$

Numerical results



Towards an inhomogeneous model

[B.D./Wolfram, '15]

More realistic models should depend on additional, independent variables, e.g.

- leadership 'strength'
- space
- ...

$\rightsquigarrow$ inhomogeneous Boltzmann-type equation for $f = f(x, w, t)$

$$\frac{\partial}{\partial t} f + \mathrm{div}_x(\Phi(x,w)f) = \frac{1}{\tau}\mathcal{Q}(f,f)$$

$\rightarrow$ choice of the field $\Phi = \Phi(x, w)$ which describes the opinion flux plays a crucial role

$\rightarrow$ in contrast to the physical situation where $\Phi(w) = w$, suitable choices need to be made

$\rightarrow$ example: inhomogeneous Boltzmann-type equation modelling political segregation in 'The Big Sort'

Political segregation: The Big Sort

The Big Sort (Bishop/Cushion, 2008):
→ clustering of individuals who share similar political opinions
→ observed in USA over the last decades

- acclaimed in many newspapers and magazines
- former president Bill Clinton urged audiences to read the book
- but claims also met opposition from political sociologists

Inhomogeneous model II: The Big Sort

Landslide counties: counties in which either candidate won or lost by 20 percentage points or more

1976 2004

→ number of landslide counties has doubled

Inhomogeneous Boltzmann-type equation

opinion variable $w \in [-1, 1]$

additional variable: position on map $x \in \Omega \subset \mathbb{R}^2$

opinion distribution function $f = f(x, w, t)$

supporters of the parties given by marginals

$$B(x,t) = \int_{-1}^{0} f(x,w,t)\, dw, \quad R(x,t) = \int_{0}^{1} f(x,w,t)\, dw$$

opinion dynamics ($\sim$ Toscani, 2006):

$$v^* = v - \gamma K(|x-y|) P(|v-w|)(v-w) + \eta_1 D(v)$$
$$w^* = w - \gamma K(|y-x|) P(|v-w|)(w-v) + \eta_2 D(w)$$

We are lead to study

$$\frac{\partial}{\partial t} f + \mathrm{div}_x(\Phi(x,w) f) = \frac{1}{\tau} \mathcal{Q}(f,f).$$

Fokker-Planck limit

Full inhomogeneous Boltzmann-type equation difficult to handle

Idea: Derive macroscopic approximation

For $t' = \gamma t,\ x' = \gamma x$ let $g(x', w, t') = f(x, w, t)$.

Limit: $\gamma, \sigma \to 0$ with $\lambda = \sigma^2/\gamma$ fixed

$\to$ (scaled) density converges to $g(x, w, t)$, which solves

$$\frac{\partial}{\partial t} g(x,w,t) + \operatorname{div}_x(\Phi(x,w) g(x,w,t))$$
$$= \frac{\partial}{\partial w}\left(\frac{1}{\tau}\mathcal{K}(x,w,t) g(x,w,t)\right) + \frac{\lambda M(x)}{2\tau}\frac{\partial^2}{\partial w^2}\left(D^2(w) g(x,w,t)\right)$$

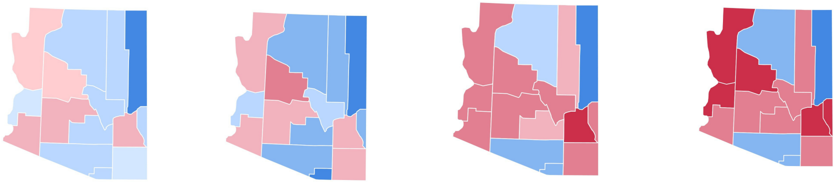
with nonlocal drift operator

$\mathcal{K}(x,w,t) = \int_{\mathcal{I}} P(|w-v|)(w-v) g(x,v,t)\,dv$

and mass $M(x) = \int g(x,v,t)\,dv$

Numerical example: The Big Sort in Arizona

Results of the US presidential elections in the state of Arizona:



1992 1996 2000 2004

Colour intensities reflect election outcome in percent:

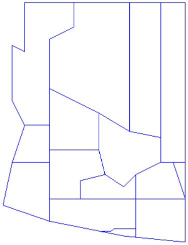
- dark blue (red) to Democrats (Republicans) 60–70%
- medium blue (red) to Democrats (Republicans) 50–60%
- light blue (red) to Democrats (Republicans) 40–50%

Numerical results: The Big Sort in Arizona

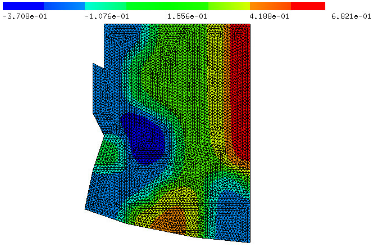
Initial distribution: $f(x,w,0)$ proportional to 1992 elections
Potential $\Phi(x,w) = \mathrm{sgn}(w)\nabla C(x)(1-w^2)^\gamma$ is obtained from 1996 election results by solving

$$C(x) + \varepsilon\Delta C(x) = f_{1996}(x)$$

Solve the Fokker-Planck on $\Omega \subset \mathbb{R}^2$ corresponding to the state Arizona, divided into 15 electoral counties:



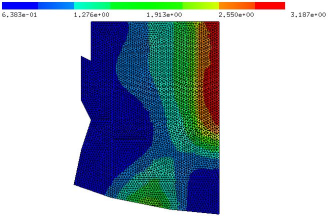
Domain Ω



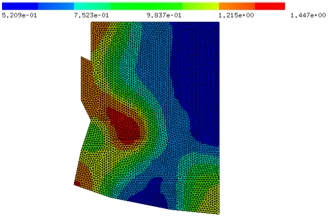
Potential $C(x)$

Numerical results: The Big Sort in Arizona

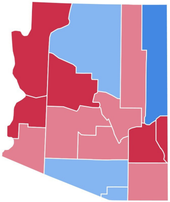
Simulation results:



Democrats



Republicans



Election results 2004

$\rightarrow$ patterns of the electoral results from 2004 are reproduced fairly well, except for the second county from the right in the Northeast (Navajo)

Summary and perspectives

Summary:

- complex systems in economics and sociology
  $\rightsquigarrow$ Boltzmann-type equations
- microscopic dynamics
- emergent behaviour
- examples:
  - wealth distribution
  - opinion dynamics under opinion leadership
  - political segregation: 'The Big Sort' in USA

Perspectives:

- better understanding of input data (micro/macro)
- asymptotic behaviour of moments
- optimal control
- optimal strategies, mean-field games
- connection to graph-based approaches