Blending Mathematical Models and Data: Algorithms, Analysis and Applications

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Enabling Quantification of
EQUIP
Uncertainty for Inverse Problems

THE UNIVERSITY OF
WARWICK

Outline

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Current State of Play

Data Everywhere

In every realm of human experience, growing in volume.

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Where Are We Now?

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Prototypical Problem Areas

- Physics (Newton's Laws, Quantum Mechanics).
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All Have Vast Amounts of Data!

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- What extra do we learn from the data?

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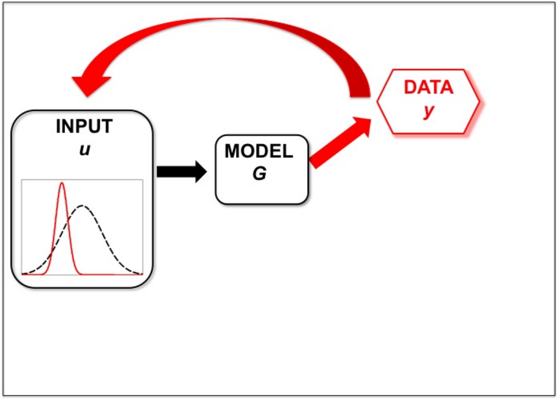
Bayes Theorem (Picture)



Model

$G : X \to \mathbb{R}^J$, X separable Banach space.

Bayes Theorem (Mathematics)

Prior

Probabilistic information about u before data is collected: $\mathbb{P}(du)$.

Likelihood

Since $y = \mathcal{G}(u) + \eta$, if $\eta \sim N(0, \Gamma)$, then $y|u \sim N(\mathcal{G}(u), \Gamma)$. The model-data misfit Φ is the negative log-likelihood:

$$\mathbb{P}(y|u) \propto \exp\big(-\Phi(u;y)\big), \quad \phi(u;y) = \frac{1}{2}\Big|\Gamma^{-1/2}\big(y - \mathcal{G}(u)\big)\Big|^2.$$

Posterior

Probabilistic information about u after data is collected:

$$\mathbb{P}(du|y) \propto \exp\big(-\Phi(u;y)\big)\mathbb{P}(du).$$

Algorithms For Bayesian Inversion

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- Approximation by high dimensional integration: Harmonic analysis, sparse integration [5].

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Example 1: Piecewise Constant Reconstruction

Forward Problem

Let $K \in \mathcal{L}(X, \mathbb{R}^J)$ for some Banach space X. Given $\sigma \in X$

$$y = K\sigma.$$

Let $\eta \in \mathbb{R}^J$ be a realization of an observational noise.

Inverse Problem

Impose prior information that σ is piecwise constant:

$$\sigma = F(u) := \sigma^+ \chi_{\{u \geq 0\}}(x) + \sigma^- \chi_{\{u<0\}}(x).$$

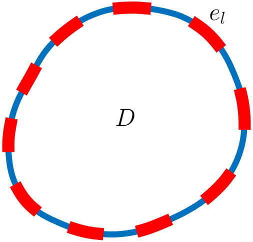
Given $y \in \mathbb{R}^J$, find u such that

$$y = KF(u) + \eta.$$

Example 2: Electrical Impedance Tomography

M. Dunlop and A.M. Stuart
Bayesian formulation of EIT.
arXiv:1509.03136
Inverse Problems and Imaging, Submitted, 2015.

- Apply currents I_ℓ on e_ℓ, $\ell = 1, \ldots, L$.
- Induces voltages Θ_ℓ on e_ℓ, $\ell = 1, \ldots, L$.
- We have an Ohm's law $\Theta = R(\sigma)I$.
- Find conductivity σ from measurements (I, Θ).



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- The UK has world-leading expertise in many of the constituent areas of the mathematical sciences.
- An essential barrier, which the ATI can help overcome, is to bring together this expertise to effectively identify significant intellectual challenges which underpin different problem classes.
- In doing so define data science as a discipline.

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MATLAB files and book chapters freely available:
`http://tiny.cc/damat`
`http://arxiv.org/abs/1506.07825`