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Compatible finite element methods for numerical weather prediction

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Compatible finite element spaces

$$
\begin{tikzcd}
H^1 \arrow[r, "\nabla^\perp=(-\partial_y,\partial_x)"] \arrow[d, "\pi_0"] & H(\mathrm{div}) \arrow[r, "\nabla\cdot"] \arrow[d, "\pi_1"] & L^2 \arrow[d, "\pi_2"] \\
\mathbb{V}^0 \arrow[r, "\nabla^\perp"] & \mathbb{V}^1 \arrow[r, "\nabla\cdot"] & \mathbb{V}^2
\end{tikzcd}
$$

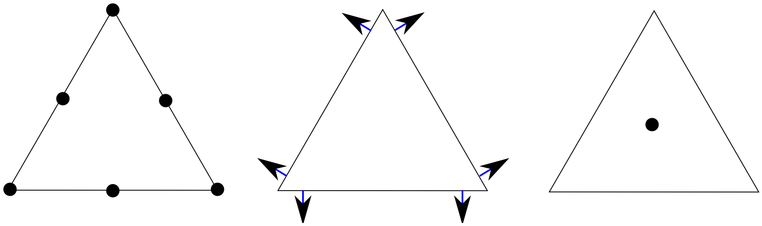
Requirements

1. $\nabla\cdot$ maps from $\mathbb{V}^1$ onto $\mathbb{V}^2$, and $\nabla^\perp$ maps from $\mathbb{V}^0$ onto kernel of $\nabla\cdot$ in $\mathbb{V}^1$.
2. Commuting, bounded surjective projections π_i exist.

Book: "Finite Element Exterior Calculus" by Doug Arnold.

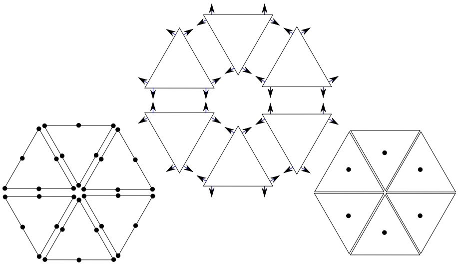
Compatible FE spaces

$$\underbrace{\mathbb{V}_0 = P2}_{\text{Quadratic, Continuous}} \xrightarrow{\nabla^\perp} \underbrace{\mathbb{V}_1 = BDM1}_{\text{Linear, Continuous normals}} \xrightarrow{\nabla\cdot} \underbrace{\mathbb{V}_2 = P0}_{\text{Constant, Discontinuous}}$$



Compatible FE spaces in 2D

$$\underbrace{\mathbb{V}_0 = P2}_{\text{Quadratic, Continuous}} \xrightarrow{\nabla^{\perp}} \underbrace{\mathbb{V}_1 = BDM1}_{\text{Linear, Continuous normals}} \xrightarrow{\nabla\cdot} \underbrace{\mathbb{V}_2 = P0}_{\text{Constant, Discontinuous}}$$



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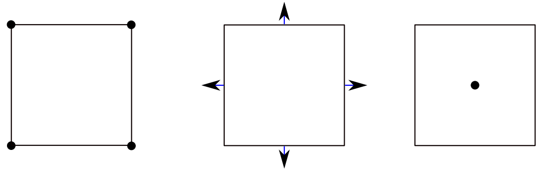
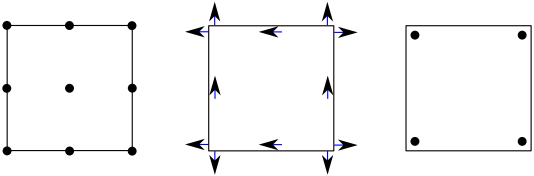
Compatible FE spaces

$$\underbrace{\mathbb{V}_0 = Q1}_{\text{Bilinear Continuous}} \xrightarrow{\nabla^\perp} \underbrace{\mathbb{V}_1 = RT0}_{\text{Constant/Linear, Cont. normals}} \xrightarrow{\nabla\cdot} \underbrace{\mathbb{V}_2 = Q0_{DG}}_{\text{Constant, Discontinuous}}$$



Compatible FE spaces

$$\underbrace{\mathbb{V}_0 = Q2}_{\text{Biquadratic Continuous}} \xrightarrow{\nabla^\perp} \underbrace{\mathbb{V}_1 = RT1}_{\text{Bilinear/Biquadratic, Cont. normals}} \xrightarrow{\nabla\cdot} \underbrace{\mathbb{V}_2 = Q1_{DG}}_{\text{Bilinear, Discontinuous}}$$



Helmholtz decomposition

Helmholtz decomposition

For a (here, boundaryless) domain Ω, any $\boldsymbol{u}^\delta \in \mathbb{V}_1$ can be uniquely written as

$$\boldsymbol{u}^\delta = \nabla^\perp \psi^\delta + \mathfrak{h}^\delta + \tilde{\nabla}\phi^\delta,$$

with $\psi^\delta \in \mathbb{V}_0$, $\mathfrak{h}^\delta \in \mathfrak{h}_1$, $\phi^\delta \in \mathbb{V}_2$,

$$\mathfrak{h}_1 = \left\{ \mathfrak{h}^\delta \in \mathbb{V}_1 : \nabla \cdot \mathfrak{h}^\delta = \tilde{\nabla}^\perp \cdot \mathfrak{h}^\delta = 0 \right\},$$

$$\left\langle \boldsymbol{w}^\delta, \tilde{\nabla}\phi^\delta \right\rangle = -\left\langle \nabla \cdot \boldsymbol{w}^\delta, \phi^\delta \right\rangle, \quad \forall \boldsymbol{w}^\delta \in \mathbb{V}_1,$$

$$\left\langle \gamma^\delta, \tilde{\nabla}^\perp \cdot \boldsymbol{u}^\delta \right\rangle = -\left\langle \nabla^\perp \gamma^\delta, \boldsymbol{u}^\delta \right\rangle, \quad \forall \gamma^\delta \in \mathbb{V}_0.$$

Linear, rotating shallow water equations (f plane)

$$
\boldsymbol{u}_t + \underbrace{f\boldsymbol{u}^{\perp}}_{\text{Coriolis}} + \underbrace{g\nabla h}_{\text{Pressure gradient}} = 0,
$$

$$
h_t + \underbrace{H_0\nabla\cdot\boldsymbol{u}}_{\text{Mass flux}} = 0.
$$

Mixed finite element discretisation: seek $\boldsymbol{u}^\delta \in \mathbb{V}_1$, $h^\delta \in \mathbb{V}_2$ s.t.

$$
\begin{aligned}
\langle \boldsymbol{w}^\delta, \boldsymbol{u}^\delta_t\rangle + f\langle \boldsymbol{w}^\delta, \left(\boldsymbol{u}^\delta\right)^\perp\rangle - \langle \nabla\cdot\boldsymbol{w}^\delta, gh^\delta\rangle &= 0, \quad \forall \boldsymbol{w}^\delta \in \mathbb{V}_1, \\
\langle \gamma^\delta, h^\delta_t\rangle + \langle \gamma^\delta, H_0\nabla\cdot\boldsymbol{u}^\delta\rangle &= 0, \quad \forall \gamma^\delta \in \mathbb{V}_2.
\end{aligned}
$$

For weather forecasting applications, require:

- Inf-sup condition for pressure gradient term (classical)
- Geostrophic balance condition (steady states divergence-free).
- No spurious inertial oscillations

Proposition (Geostrophic balance condition, CJC and Shipton (2012))

For all divergence-free $\boldsymbol{u}^\delta$, there exists $h^\delta \in \mathbb{V}_2$ such that $(\boldsymbol{u}^\delta, h^\delta)$ is a steady state solution of the linear f-plane equations.

Proof.

Take $\psi^\delta \in \mathbb{V}_0$ such that $\nabla^\perp \psi^\delta = \boldsymbol{u}^\delta$, and define h^δ by $\langle \gamma^\delta, f\psi^\delta \rangle = \langle \gamma^\delta, gh^\delta \rangle$, $\forall \gamma^\delta \in \mathbb{V}_2$. Then,

$$
\begin{aligned}
\langle \boldsymbol{w}^\delta, \boldsymbol{u}^\delta_t \rangle &= -\langle \boldsymbol{w}^\delta, f\left(\boldsymbol{u}^\delta\right)^\perp \rangle + \langle \nabla \cdot \boldsymbol{w}^\delta, gh^\delta \rangle, \\
&= \langle \boldsymbol{w}^\delta, f\nabla\psi^\delta \rangle + \langle \nabla \cdot \boldsymbol{w}^\delta, gh^\delta \rangle, \\
&= -\langle \nabla \cdot \boldsymbol{w}^\delta, f\psi^\delta \rangle + \langle \nabla \cdot \boldsymbol{w}^\delta, gh^\delta \rangle = 0, \ \forall \boldsymbol{w}^\delta \in \mathbb{V}_1.
\end{aligned}
$$

$\square$

Inertial oscillations

$$\boldsymbol{u}_t + \underbrace{f\boldsymbol{u}^\perp}_{\text{Coriolis}} + \underbrace{g\nabla h}_{\text{Pressure gradient}} = 0,$$

$$h_t + \underbrace{H_0\nabla\cdot\boldsymbol{u}}_{\text{Mass flux}} = 0.$$

Absence of spurious inertial oscillations follows from having harmonic functions of correct dimension. Natale, Shipton and Cotter (2016).

Nonlinear shallow water equations

Advective form:

$$\boldsymbol{u}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + g\nabla D = 0,$$
$$D_t + \nabla\cdot(\boldsymbol{u}D) = 0.$$

Vector-invariant form:

$$\boldsymbol{u}_t + qD\boldsymbol{u}^\perp + \nabla\left(gD + \frac{1}{2}|\boldsymbol{u}|^2\right) = 0, \quad D_t + \nabla\cdot(\boldsymbol{u}D) = 0,$$

$$\text{where } q = \frac{\nabla^\perp\cdot\boldsymbol{u} + f}{D},$$

$$\implies q_t + \boldsymbol{u}\cdot\nabla q = 0 \implies \frac{\partial}{\partial t}\int_\Omega q^\alpha D\,\mathrm{d}x = 0\ \forall\alpha.$$

Nonlinear shallow water equations

$$\boldsymbol{u}_t + qD\boldsymbol{u}^\perp + \nabla\left(gD + \frac{1}{2}|\boldsymbol{u}|^2\right) = 0, \quad D_t + \nabla\cdot(\boldsymbol{u}D) = 0,$$

$$\text{where } q = \frac{\nabla^\perp\cdot\boldsymbol{u} + f}{D}.$$

Discretisation: $\boldsymbol{u} \in \mathbb{V}_1$, $D \in \mathbb{V}_2$, and define $\boldsymbol{F} \in \mathbb{V}_1$, $q \in \mathbb{V}_0$ with

$$\begin{aligned}
\langle \boldsymbol{w}, \boldsymbol{F}\rangle - \langle \boldsymbol{w}, \boldsymbol{u}D\rangle &= 0, \quad \forall \boldsymbol{w} \in \mathbb{V}_1, \\
\langle \gamma, qD\rangle - \langle -\nabla^\perp\gamma, \boldsymbol{u}\rangle - \langle \gamma, f\rangle &= 0, \quad \forall \gamma \in \mathbb{V}_0, \\
\langle \boldsymbol{w}, \boldsymbol{u}_t\rangle + \langle \boldsymbol{w}, q\boldsymbol{F}^\perp\rangle - \langle \nabla\cdot\boldsymbol{w}, gD + \frac{1}{2}|\boldsymbol{u}|^2\rangle &= 0, \quad \forall \boldsymbol{w} \in \mathbb{V}_1, \\
\langle \phi, D_t + \nabla\cdot\boldsymbol{F}\rangle &= 0, \quad \forall \phi \in \mathbb{V}_2.
\end{aligned}$$

D equation is satisfied pointwise!

(Almost) Poisson structure

$$\dot{F} + \{F, H\} = 0,$$

with

$$\begin{aligned}
\{F, G\} &= \int_\Omega \frac{\delta F}{\delta \boldsymbol{u}} \cdot \frac{\delta G}{\delta \boldsymbol{u}}^\perp q \, \mathrm{d}x \\
&\quad + \int_\Omega \nabla \cdot \frac{\delta F}{\delta \boldsymbol{u}} \frac{\delta G}{\delta D} \, \mathrm{d}x - \int_\Omega \nabla \cdot \frac{\delta G}{\delta \boldsymbol{u}} \frac{\delta F}{\delta D} \, \mathrm{d}x, \\
H &= \frac{1}{2} \int_\Omega D \|\boldsymbol{u}\|^2 + g D^2 \, \mathrm{d}x.
\end{aligned}$$

McRae and CJC (2014)

Energy conservation

$$
\begin{aligned}
E &= \int \frac{1}{2} D|\boldsymbol{u}|^2 + \frac{1}{2} g D^2 \,\mathrm{d}x, \\
\dot{E} &= \langle \frac{1}{2}|\boldsymbol{u}|^2 + gD, D_t \rangle + \underbrace{\langle D\boldsymbol{u}, \boldsymbol{u}_t \rangle}_{=\langle \boldsymbol{F}, \boldsymbol{u}_t \rangle}, \\
&= \langle \frac{1}{2}|\boldsymbol{u}|^2 + gD, -\nabla\cdot\boldsymbol{F} \rangle \\
&\quad + \langle \boldsymbol{F}, -q\boldsymbol{F}^{\perp} \rangle + \langle \nabla\cdot\boldsymbol{F}, \frac{1}{2}|\boldsymbol{u}|^2 + gD \rangle = 0.
\end{aligned}
$$

Or from antisymmetry of the bracket.

Enstrophy conservation

$$
\begin{aligned}
Z &= \int D q^2 \,\mathrm{d}x, \\
\dot{Z} &= -\langle \dot{D}, q^2 \rangle + 2\langle q, (qD)_t \rangle, \\
&= -\langle \dot{D}, q^2 \rangle - 2\langle \nabla^\perp q, \boldsymbol{u}_t \rangle, \\
&= \langle \nabla \cdot \boldsymbol{F}, q^2 \rangle + 2\langle \nabla^\perp q, q\boldsymbol{F}^\perp \rangle \\
&\quad - 2\langle \underbrace{\nabla \cdot \nabla^\perp q}_{=0}, q\boldsymbol{F}^\perp \rangle, \\
&= -\langle \boldsymbol{F}, \nabla q^2 \rangle + \langle \nabla q^2, \boldsymbol{F} \rangle = 0.
\end{aligned}
$$

Or by showing Z is a Casimir of the bracket.

Implied PV conservation

$$\langle \boldsymbol{w}, \boldsymbol{u}_t \rangle + \langle \boldsymbol{w}, q\boldsymbol{F}^\perp \rangle - \langle \nabla \cdot \boldsymbol{w}, gD + \frac{1}{2}|\boldsymbol{u}|^2 \rangle = 0, \quad \forall \boldsymbol{w} \in \mathbb{V}_1.$$

For $\gamma \in \mathbb{V}_0$, take $\boldsymbol{w} = -\nabla^\perp \gamma$,

$$\langle -\nabla^\perp \gamma, \boldsymbol{u}_t \rangle - \langle \nabla \gamma, q\boldsymbol{F} \rangle = 0, \quad \forall \gamma \in \mathbb{V}_0.$$

Recall definition of q:

$$\langle \gamma, qD \rangle = \langle -\nabla^\perp \gamma, \boldsymbol{u} \rangle + \langle \gamma, f \rangle, \quad \forall \gamma \in \mathbb{V}_0.$$

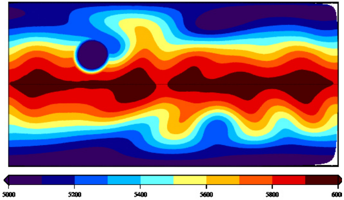
Hence we get the implied equation for diagnostic q,

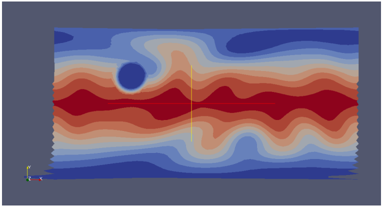
$$\langle \gamma, (qD)_t \rangle - \langle \nabla \gamma, q\boldsymbol{F} \rangle = 0, \quad \forall \gamma \in \mathbb{V}_0,$$

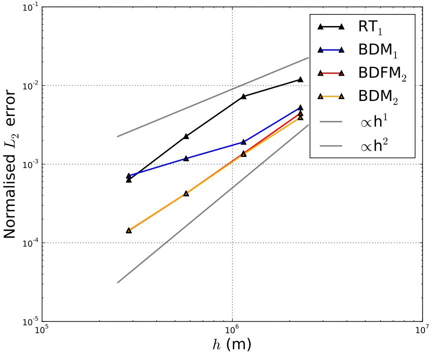
$$\text{i.e. } \langle \gamma, (qD)_t + \nabla \cdot (q\boldsymbol{F}) \rangle = 0, \quad \forall \gamma \in \mathbb{V}_0.$$

Testcases

Mountain test case (Grid 5, 46080 DOFs).







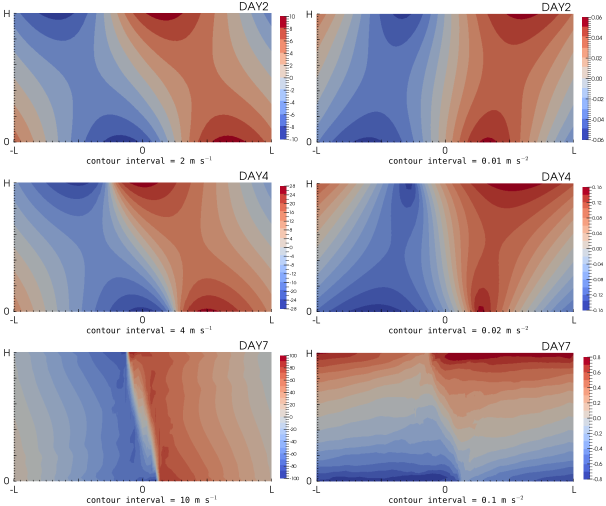
Normalised L_2 error
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{5}
10^{6}
10^{7}
h (m)
RT_1
BDM_1
$BDFM_2$
BDM_2
$\propto h^1$
$\propto h^2$

Extensions for nonlinear SWE

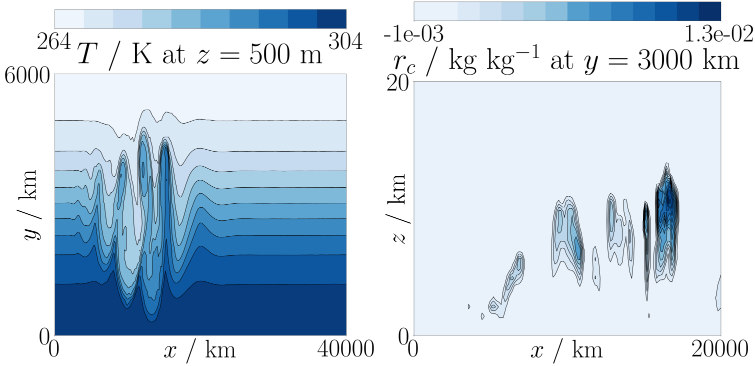
- Variational discretisation from Hamilton's principle (incompressible): Natale and CJC (2014)
- Higher-order upwinding with consistent PV transport: CJC, Gibson and Shipton (2018)
- Energy-enstrophy conservation with boundaries: Bauer and CJC (2018)
- Mimetic spectral elements: Lee, Palha and Gerritsma (2018)
- Energy-conserving upwinding: Wimmer, CJC and Bauer (2019)
- Spline-based finite element spaces: Eldred, Dubos and Kritsikis (2019)

Extensions to 3D (and vertical slice) equations

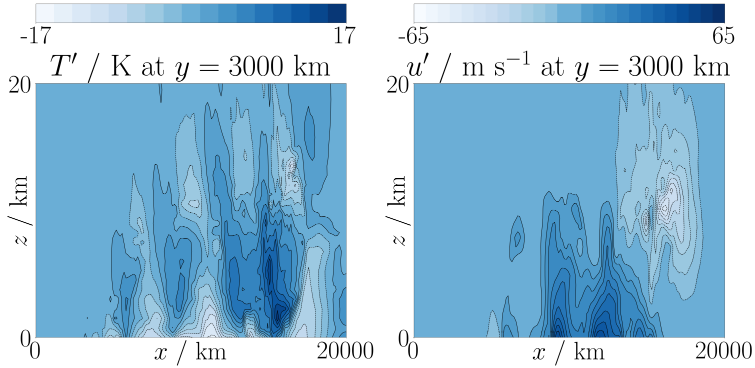
- Frontogenesis test cases: Yamazaki et al (2017)
- Tensor product spaces including for temperature: Natale, Shipton and Cotter (2016), Melvin et al (2018)
- Lowest order spaces: Bendall, Cotter and Shipton (2019), Melvin et al (2019)
- Firedrake implementation: Gibson et al (2019a)
- Hybridised solvers for 3D equations: Gibson et al (2019b)
- Moisture: Bendall et al (2019)


DAY2
H
0
-L
0
L
contour interval = 2 m s^{-1}
DAY2
contour interval = 0.01 m s^{-2}
DAY4
contour interval = 4 m s^{-1}
DAY4
contour interval = 0.02 m s^{-2}
DAY7
contour interval = 10 m s^{-1}
DAY7
contour interval = 0.1 m s^{-2}

Moist baroclinic wave



Moist baroclinic wave



Summary

- Discrete de Rham complex gives Helmholtz decomposition that leads to correct coupling between slow and fast waves in linearised system
- Energy-enstrophy conservation obtained from combining compatible finite element spaces and almost Poisson structure.
- Extensions are innovating structure-preserving methods as well as practical simulations

Vertical discretisation

Given 2D finite element spaces $(U_0 \stackrel{\nabla^\perp}{\rightarrow} U_1 \stackrel{\nabla\cdot}{\rightarrow} U_2)$ and 1D finite element spaces $(V_0 \stackrel{\partial_x}{\rightarrow} V_1)$, we can generate a product in three dimensions:

$$W_0 \xrightarrow{\nabla} W_1 \xrightarrow{\nabla\times} W_2 \xrightarrow{\nabla\cdot} W_3,$$

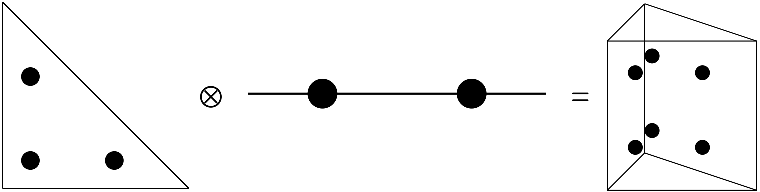
where

$$\begin{aligned}
W_0 &:= U_0 \otimes V_0, \\
W_1 &:= \mathtt{HCurl}(U_0 \otimes V_1) \oplus \mathtt{HCurl}(U_1 \otimes V_0), \\
W_2 &:= \mathtt{HDiv}(U_1 \otimes V_1) \oplus \mathtt{HDiv}(U_2 \otimes V_0), \\
W_3 &:= U_2 \otimes V_1,
\end{aligned}$$

with $W_0 \subset H^1, W_1 \subset H(\text{curl}), W_2 \subset H(\text{div}), W_3 \subset L^2$

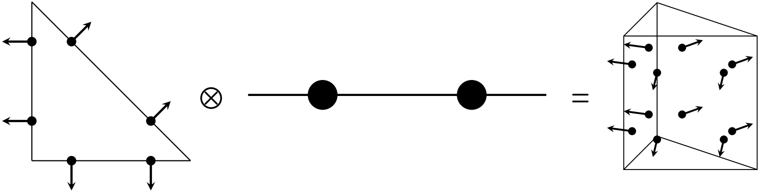
3D spaces

Density space



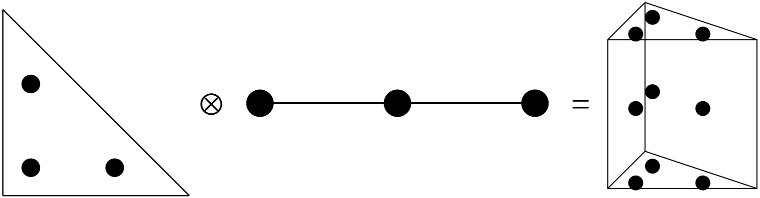
3D spaces

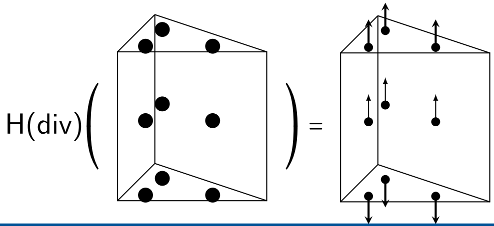
Horizontal part of velocity space



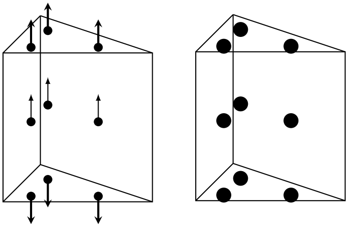
3D spaces

Vertical part of velocity space





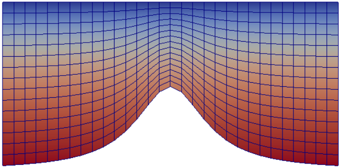
Where to store temperature?



Charney-Phillips for FEM

We use the same node locations for temperature as the vertical part of velocity.

No spurious hydrostatic modes



$0 = -\theta \frac{\partial \Pi}{\partial z} - g$ becomes
$\int_\Omega \nabla \cdot (\theta \boldsymbol{w}) \Pi \, \mathrm{d}x + \int_\Omega \boldsymbol{w} \cdot \hat{\boldsymbol{z}} g \, \mathrm{d}x = 0, \ \forall \boldsymbol{w} \in \mathbb{W}_2^v,$
with $\Pi \in \mathbb{W}_3$, $\theta \in \mathbb{W}_2^\theta$.

Proposition (Hydrostatic modes, Natale and CJC (2016))

With no slip boundary conditions, and a columnar mesh, there is a one-to-one mapping between θ and Π (up to a constant). The mapping is unique if there is a (weak) Dirichlet boundary condition for p on the top surface.

Proof.

By standard Brezzi mixed FEM conditions on $\mathbb{W}_2^\theta$ and $\mathbb{W}_3$. $\square$

The challenge

Finite element space choice comes from linear considerations. Need good schemes for the nonlinear equations under this constraint.

Some issues:

- Stable, accurate advection term in momentum equation. ✓
- Stable, accurate advection scheme for temperature space. ✓
- Viscosity/diffusion terms. ✓
- Bounded advection for temperature space tracers. ✓
- Nonlinear pressure gradient term (for compressible). ✓
- Efficient solver for semi-implicit timestepping. ✓

Hybridisation

Replace W_2 with $\bar{W}_2$, which has the same basis functions but without inter-element continuity. Enforce inter-element continuity with Lagrange multipliers integrated over element edges.

Having analytically eliminated p, seek $\boldsymbol{u} \in \bar{W}_2$, $\Lambda \in T(W_2)$ such that:

$$\int_\Omega \boldsymbol{w}\cdot\boldsymbol{u} + \frac{c^2\Delta t^2}{4}\nabla\cdot\boldsymbol{w}\nabla\cdot\boldsymbol{u}\,\mathrm{d}x + \int_\Gamma [[\boldsymbol{w}\cdot\boldsymbol{n}]]\lambda\,\mathrm{d}s = \tilde{\mathcal{R}}_{\boldsymbol{u}}[\boldsymbol{w}], \quad \forall \boldsymbol{w}\in\bar{W}_2$$

$$\int_\Gamma \mu[[\boldsymbol{u}\cdot\boldsymbol{n}]]\,\mathrm{d}s = 0, \quad \forall\mu\in T(W_2).$$

Matrix equations:

$$\bar{H}\hat{\boldsymbol{u}} + L^T\hat{\boldsymbol{\lambda}} = \bar{R}_{\boldsymbol{u}},$$
$$L\hat{\boldsymbol{u}} = 0.$$

Becomes $L\bar{H}^{-1}L^T\hat{\boldsymbol{\lambda}} = \mathrm{RHS}_\lambda$.

```
BrokenV1 = FunctionSpace(mesh, BrokenElement(V1_elt))
TraceV1 = FunctionSpace(mesh, TraceElement(V1_elt))
```

```
Ainv = assemble(aWavelhs,inverse=True).M.handle
```

```
tmp = Ainv.matMult(B)
tmp = C.matMult(tmp)
S = D.copy()
S.axpy(-1, tmp)
```

Nonlinear pressure gradient term

Pressure gradient term

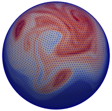
In θ-Π formulation of compressible equations, the pressure gradient term is $\theta\nabla\Pi$, where Π is a function of ρ and θ.

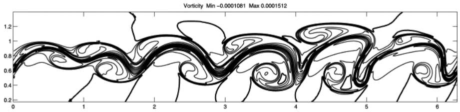
- θ is discontinuous in the horizontal.
- Π is discontinous (as function of $\rho \in \mathbb{W}_3$).

$$\int_\Omega \boldsymbol{w}\cdot\boldsymbol{g}\,\mathrm{d}x = \int_\Omega \nabla_h\cdot(\theta\boldsymbol{w})\Pi\,\mathrm{d}x - \int_{\partial\Omega}\boldsymbol{n}\cdot\boldsymbol{w}[[\Pi]]\{\theta\}\,\mathrm{d}S, \quad \forall \boldsymbol{w}\in\mathbb{V}_2.$$

On cuboid elements, after lumping the mass matrix, this is the same as the Met Office Unified Model (ENDGAME) discretisation.

Application to nonlinear shallow water equations

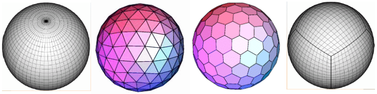




- FEniCS tools for implementation on the sphere Rognes, Ham, CJC and McRae GMD (2013).
- Energy-enstrophy conserving formulation McRae and CJC QJRMS (2014).
- Connections with finite element exterior calculus CJC and Thuburn JCP (2014).
- Dual-grid formulation with PV conservation Thuburn and CJC JCP (in press).

Staggered finite difference methods

We sought a staggered FD scheme on pseudouniform grids:



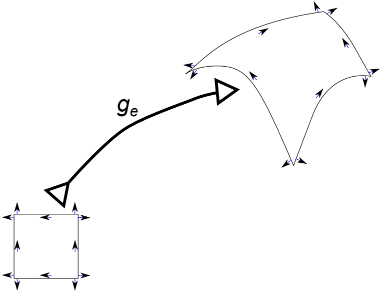
What goes wrong:

- On triangular grids: spurious inertia-gravity wave modes.
- On quadratilateral grids: need nonorthogonal grid, loss of consistency in Coriolis term (if steady geostrophic mode property required).
- On dual-icosahedral (hex/pent) grids: loss of consistency in Coriolis term (if steady geostrophic mode property required).

non-orthogonal: Thuburn and CJC. SIAM J. Sci. Comp. (2012)

loss of consistency: Thuburn, CJC, and Dubos. GMD (2014)

Construction of $\mathbb{S}$



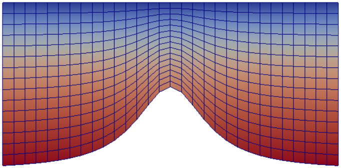
$\boldsymbol{g}_e : e' \to e$, $\boldsymbol{x} = \boldsymbol{g}_e(\boldsymbol{x}')$.

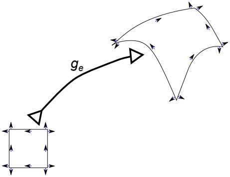
The Piola transformation $\boldsymbol{u}' \mapsto \boldsymbol{u}$:

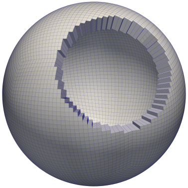
$$\boldsymbol{u} \circ \boldsymbol{g}_e = \frac{1}{\det \frac{\partial \boldsymbol{g}_e}{\partial \boldsymbol{x}'}} \frac{\partial \boldsymbol{g}_e}{\partial \boldsymbol{x}'} \boldsymbol{u}'$$

$$\boldsymbol{u}' \cdot \boldsymbol{n}' \, \mathrm{d}x' = \boldsymbol{g}_e^*(\boldsymbol{u} \cdot \boldsymbol{n} \, \mathrm{d}x)$$

Approximation theory







- On non-affine elements, compatible finite element spaces become non-polynomial.
- Cubed sphere quadrilateral meshes, higher-order sphere triangle meshes, topography, 3D global meshes all call for non-affine elements.

How does this impact approximation properties?

Approximation theory

- Holst and Stern, FoCM (2012) showed that approximation order is not reduced provided that mesh is obtained from piecewise C^∞ global transformation from an affine mesh.
- The mapping $G : \Omega \to \mathbb{R}^4$ given by

$$\boldsymbol{x} = (x, y, z) \mapsto G(\boldsymbol{x}) = \left(\frac{x}{|\boldsymbol{x}|}, \frac{y}{|\boldsymbol{x}|}, \frac{z}{|\boldsymbol{x}|}, |\boldsymbol{x}| \right),$$

  defines a domain that can be meshed using affine elements.
- For (piecewise C^∞) terrain-following meshes, C can be composed with the global mapping to the spherical annulus.
- Similar construction works for cubed sphere.

A. Natale, Shipton and CJC (Dynamics and Statistics of the Climate System, 2016)

Approximation theory